

ON SUBSHIFT PRESENTATIONS

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ABSTRACT. We consider partitioned graphs, by which we mean finite strongly connected directed graphs with a partitioned edge set $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$. With additionally given a relation \mathcal{R} between the edges in \mathcal{E}^- and the edges in \mathcal{E}^+ , and denoting the vertex set of the graph by \mathfrak{P} , we speak of an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. From \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we construct semigroups (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that we call \mathcal{R} -graph semigroups. We write a list of conditions on a topologically transitive subshift with Property (A) that together are sufficient for the subshift to have an \mathcal{R} -graph semigroup as its associated semigroup. Generalizing previous constructions, we describe a method of presenting subshifts by means of suitably structured labelled directed graphs $(\mathcal{V}, \Sigma, \lambda)$ with vertex set \mathcal{V} , edge set Σ , and a label map that assigns to the edges in Σ labels in an \mathcal{R} -graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. We denote the presented subshift by $X(\mathcal{V}, \Sigma, \lambda)$ and call $X(\mathcal{V}, \Sigma, \lambda)$ an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation.

We introduce a Property (B) of subshifts that describes a relationship between contexts of admissible words of a subshift, and we introduce a Property (C) of subshifts, that is stronger than Property (B), that in addition describes a relationship between the past and future contexts and the context of admissible words of a subshift. Property (B) and Property (C) are invariants of topological conjugacy.

We consider presentations of subshifts, in which every symbol of the alphabet has a future, that is compatible with its entire past context. Such subshift presentations we call right instantaneous presentations. We introduce a Property *RI* of subshifts, and we prove that property *RI* is a necessary and sufficient condition for the subshift to be topologically conjugate to a right instantaneous presentation. We consider also presentations of subshifts, in which every symbol of the alphabet has a future, that is compatible with its entire past context, and also a past that is compatible with its entire future context. Such subshift presentations we call bi-instantaneous presentations. The simultaneous presence of Property *RI* in a subshift and in its inverse is a necessary and sufficient condition for the subshift to be topologically conjugate to a bi-instantaneous presentation.

We introduce a notion of strong instantaneity. Under an assumption on the structure of the \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we show for strongly instantaneous subshifts with Property (A) and associated semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, that Property (C) is necessary and sufficient for the existence of an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation, to which the subshift is topologically conjugate.

Let $\Pi > 1$ and denote by $S(\Pi)$ the graph inverse semigroup of the directed graph with Π vertices, that has Π^2 edges from every vertex to every vertex. We construct a subshift with property (A) and associated semigroup $S(\Pi)$ that does not have Property (C).

We associate to the finite directed labelled graphs $(\mathcal{V}, \Sigma, \lambda)$ topological Markov chains and Markov codes, and we derive an expression for the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$ in terms of the zeta functions of the topological Markov shifts and the generating functions of the Markov codes.

1. INTRODUCTION

Let there be given a finite strongly connected directed graph with vertex set \mathfrak{P} and edge set \mathcal{E} . We denote the initial vertex of an edge $e \in \mathcal{E}$ by $s(e)$ and its final vertex by $t(e)$. Assume also given a partition

$$\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+.$$

We call the structure $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ a partitioned graph. We set

$$\begin{aligned} \mathcal{E}^-(q, r) &= \{e^- \in \mathcal{E}^- : s(e^-) = q, t(e^-) = r\}, \\ \mathcal{E}^+(q, r) &= \{e^+ \in \mathcal{E}^+ : s(e^+) = r, t(e^+) = q\}, \quad q, r \in \mathfrak{P}. \end{aligned}$$

We assume that $\mathcal{E}^-(q, r) \neq \emptyset$ if and only if $\mathcal{E}^+(q, r) \neq \emptyset$, $q, r \in \mathfrak{P}$. Let there further be given relations

$$\mathcal{R}(q, r) \subset \mathcal{E}^-(q, r) \times \mathcal{E}^+(q, r), \quad q, r \in \mathfrak{P},$$

and set

$$\mathcal{R} = \bigcup_{q, r \in \mathfrak{P}} \mathcal{R}(q, r).$$

The structure that is given by these data, and for which we use the notation $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, we call an \mathcal{R} -graph. From an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we construct a semigroup (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that contains idempotents $\mathbf{1}_p, p \in \mathfrak{P}$, and that has \mathcal{E} as a generating set. Besides $\mathbf{1}_p^2 = \mathbf{1}_p, p \in \mathfrak{P}$, the defining relations of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ are:

$$\begin{aligned} \mathbf{1}_q e^- &= e^- \mathbf{1}_r = e^-, \quad e^- \in \mathcal{E}^-(q, r), \\ \mathbf{1}_r e^+ &= e^+ \mathbf{1}_q = e^+, \quad e^+ \in \mathcal{E}^+(q, r), \quad q, r \in \mathfrak{P}, \\ f^- g^+ &= \begin{cases} \mathbf{1}_q, & \text{if } (f^-, g^+) \in \mathcal{R}(q, r), \\ 0, & \text{if } (f^-, g^+) \notin \mathcal{R}(q, r), \end{cases} \quad f^- \in \mathcal{E}^-(q, r), g^+ \in \mathcal{E}^+(q, r), \quad q, r \in \mathfrak{P}, \end{aligned}$$

and

$$\mathbf{1}_q \mathbf{1}_r = 0, \quad q, r \in \mathfrak{P}, q \neq r.$$

We call $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ an \mathcal{R} -graph semigroup. The graph inverse semigroups (the generalized polycyclic semigroups, see [AH] and [L, Section 10.7] and compare [CK]) are a special case of \mathcal{R} -graph semigroups: The graph inverse semigroup of a finite directed graph with vertex set \mathfrak{P} and edge set \mathcal{E}° is obtained by taking a copy of the graph $(\mathfrak{P}, \mathcal{E}^\circ)$ with vertex set \mathfrak{P} and edge set $\mathcal{E}^- = \{e^- : e \in \mathcal{E}^\circ\}$ and a copy $(\mathfrak{P}, \mathcal{E}^+)$ of the reversed graph of $(\mathfrak{P}, \mathcal{E}^\circ)$ and by constructing the \mathcal{R} -graph semigroup of the partitioned graph $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ with the relations

$$\mathcal{R}(q, r) = \{(e^-, e^+) : e \in \mathcal{E}^\circ, s(e) = q, t(e) = r\}, \quad q, r \in \mathfrak{P}.$$

The Dyck inverse monoids (the polycyclic monoids) $\mathcal{D}_N, N > 1$, [NP] are obtained in this way from the one-vertex graph with N loops. For the \mathcal{R} -graph semigroups that are obtained from a one-vertex graph see also [HK, Section 4].

For a semigroup (with zero) \mathcal{S} , and for $F \in \mathcal{S}$ set

$$\Gamma(F) = \{(G^-, G^+) \in \mathcal{S} \times \mathcal{S} : G^- F G^+ \neq 0\},$$

and

$$[F] = \{F' \in \mathcal{S} : \Gamma(F') = \Gamma(F)\}.$$

The set $[\mathcal{S}] = \{[F] : F \in \mathcal{S}\}$ with the product given by

$$[G][H] = [GH], \quad G, H \in \mathcal{S},$$

is a semigroup. In section 2 we write a list of conditions on a semigroup (with zero) \mathcal{S} that together are necessary and sufficient for the semigroup to be an \mathcal{R} -graph semigroup, such that the projection of \mathcal{S} onto $[\mathcal{S}]$ is an isomorphism.

In symbolic dynamics one studies subshifts (X, S_X) , where X a shift invariant closed subset of the shift space $\Sigma^{\mathbb{Z}}$ and S_X is the restriction of the shift on $\Sigma^{\mathbb{Z}}$ to X . For an introduction to the theory of subshifts see [Ki] and [LM]. A property (A) of subshifts was introduced in [Kr2], and for a subshift X with property (A) a semigroup (with zero) $\mathcal{S}(X)$ was constructed that is invariantly associated to X .

In section 3 we translate all but the last of the conditions on the list of Section 2 into conditions on a subshift that are invariant under topological conjugacy. For the last condition we obtain a possibly stronger version for subshifts, that is also invariant under topological conjugacy. For a subshift X with property (A) these conditions together imply that $\mathcal{S}(X)$ is an \mathcal{R} -graph semigroup.

We describe now a way to present subshifts by means of \mathcal{R} -graph semigroups. We follow here closely [HIK, Section 3] where this method of presenting subshifts was introduced for the case of graph inverse semigroups of directed graphs in which every vertex has at least two incoming edges. These presentations were introduced there for the purpose of extending the criterion for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift [HI] to a wider class of target shifts. Given an \mathcal{R} -graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, denote by $\mathcal{S}_{\mathcal{R}}^-(\mathfrak{P}, \mathcal{E}^-)$ ($\mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^+)$) the subset of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that contains the non-zero elements of the subsemigroup of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that is generated by \mathcal{E}^- (\mathcal{E}^+), and consider a finite strongly connected labelled directed graph with vertex set \mathcal{V} and edge set Σ , and a labeling map λ that assign to every edge $\sigma \in \Sigma$ a label

$$(G1) \quad \lambda(\sigma) \in \mathcal{S}_{\mathcal{R}}^-(\mathfrak{P}, \mathcal{E}^-) \cup \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^+).$$

The label map λ extends to finite paths $(\sigma_i)_{1 \leq i \leq I}$, $I \in \mathbb{N}$, in the graph (\mathcal{V}, Σ) by

$$\lambda((\sigma_i)_{1 \leq i \leq I}) = \prod_{1 \leq i \leq I} \lambda(\sigma_i).$$

We denote for $\mathfrak{p} \in \mathfrak{P}$ by $\mathcal{V}(\mathfrak{p})$ the set of $V \in \mathcal{V}$ such that there is a cycle $(\sigma_i)_{1 \leq i \leq I}$, $I \in \mathbb{N}$, in the graph (\mathcal{V}, Σ) from V to V such that

$$\lambda((\sigma_i)_{1 \leq i \leq I}) = \mathbf{1}_{\mathfrak{p}}.$$

We impose the conditions (G 2 - 5):

$$(G2) \quad \mathcal{V}(\mathfrak{p}) \neq \emptyset, \quad \mathfrak{p} \in \mathfrak{P},$$

$$(G3) \quad \{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{P}\} \text{ is a partition of } \mathcal{V},$$

(G4) For $V \in \mathcal{V}(\mathfrak{p})$, $\mathfrak{p} \in \mathfrak{P}$, and for all edges e that leave V , $\mathbf{1}_{\mathfrak{p}}\lambda(e) \neq 0$, and for all edges e that enter V , $\lambda(e)\mathbf{1}_{\mathfrak{p}} \neq 0$.

(G5) For $f \in \mathcal{S}$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, such that $\mathbf{1}_{\mathfrak{q}}f\mathbf{1}_{\mathfrak{r}} \neq 0$, and for $U \in \mathcal{V}(\mathfrak{q})$, $W \in \mathcal{V}(\mathfrak{r})$, there exists a path b in the labeled directed graph $(\mathcal{V}, \Sigma, \lambda)$ from U to W such that $\lambda(b) = f$.

A finite labelled directed graph $(\mathcal{V}, \Sigma, \lambda)$, that satisfies conditions (G 1 - 5), gives rise to a subshift $X(\mathcal{V}, \Sigma, \lambda)$ that has as its language of admissible words the set of finite non-empty paths b in the graph (\mathcal{V}, λ) such that $\lambda(b) \neq 0$. We call the subshift $X(\mathcal{V}, \Sigma, \lambda)$ an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation.

Special cases of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentations are the subshifts $X_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ with alphabet \mathcal{E} and the identity map as label function. In the case of graph inverse semigroups these special cases are the Markov-Dyck shifts [M2] and in the case of the graph inverse semigroups of one-vertex graphs these special cases are the Dyck shifts [Kr1]. Add an extra loop at each vertex $\mathfrak{p} \in \mathfrak{P}$ with label $\mathbf{1}_{\mathfrak{p}}$ to obtain subshifts that in the case of graph inverse semigroups are the Markov-Motzkin shifts

[KM3], and in the case of the graph inverse semigroups of one-vertex graphs are the Motzkin shifts [I, M2].

In section 4 we introduce a Property (B) of subshifts that describes a relationship between contexts of admissible words of a subshift. In section 5 we introduce a Property (C) of subshifts, that is stronger than Property (B), that in addition describes a relationship between the past and future contexts and the context of an admissible words of a subshift. One is lead to the formulation of these properties by observing the behavior of the Dyck shifts and by abstracting their essential dynamical properties. In defining properties (B) and (C) we make reference to a presentation of the subshift, and we establish, that all presentations of the subshift have the property in question, once a given presentation has the property.

In section 6 we consider presentations of subshifts, that have the property that every symbol of the alphabet has a future, that is compatible with its entire past context. Such subshift presentations we call right instantaneous presentations. We introduce a property *RI* of subshifts, and we prove that property *RI* is a necessary and sufficient condition for the subshift to be topologically conjugate to a right instantaneous presentation. We consider also presentations of subshifts, that have the property that every symbol of the alphabet has a future, that is compatible with its entire past context, and also a past that is compatible with its entire future context. Such subshift presentations we call bi-instantaneous presentations. We introduce a property *BI* of subshifts, and we prove that property *BI* is a necessary and sufficient condition for the subshift to be topologically conjugate to a bi-instantaneous presentation. Sofic systems have bi-instantaneous presentations.

Under an assumption on the structure of the \mathcal{R} -graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ and for a subclass of the subshifts with Property *BI* and Property (A) with associated semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ we show in Section 7 that Property (C) is necessary and sufficient for the existence of an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ -presentation.

Let $\Pi > 1$ and denote by $\mathcal{S}(\Pi)$ the graph inverse semigroup of the directed graph with Π vertices, that has Π^2 edges from every vertex to every vertex. In section 8 we construct a subshift with property (A) and associated semigroup $\mathcal{S}(\Pi)$ that is not topologically conjugate to an $\mathcal{S}(\Pi)$ -presentation.

In section 9, applying methods of Keller [Ke], we associate to the finite directed labeled graph $(\mathcal{V}, \Sigma, \lambda)$ topological Markov chains and Markov codes, and we derive an expression for the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$ in terms of the zeta functions of the topological Markov shifts and the generating functions of the Markov codes.

In connection with the problem of embedding irreducible subshifts of finite type into Dyck shifts the notion of a multiplier was introduced in [HI]. A multiplier of the Dyck shift D_N , $N \in \mathbb{N}$, or more generally of an \mathcal{S}_{D_N} -presentation is an equivalence class of primitive words in the generators $\alpha(n)$, $0 \leq n \leq N$, of the Dyck inverse monoid. Here a word is called primitive if it is not the power of another word, and two primitive words are equivalent, if one is a cyclic permutation of the other. (In combinatorics these entities are called "primitive necklaces", see e.g. [BP, Section 4]). For multipliers in the case of graph inverse semigroups see [HIK]. Here the notion of a multiplier for the case of a partitioned directed graph suggests itself: A multiplier of a partitioned directed graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ (or of a $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ -presentation) is an equivalence class of pairs of cycles $((e_q^-)_{q \in \mathbb{Z}/Q\mathbb{Z}}, (e_q^+)_{q \in \mathbb{Z}/Q\mathbb{Z}})$, $Q \in \mathbb{N}$, where $(e_q^-)_{q \in \mathbb{Z}/Q\mathbb{Z}}$ is a cycle in \mathcal{E}^- and $(e_q^+)_{q \in \mathbb{Z}/Q\mathbb{Z}}$ is a cycle in \mathcal{E}^+ such that $s(e_q^-) = t(e_q^+)$, $0 \leq q < Q$, and such that for no proper divisor Q' of Q , $e_q^- = e_{q+Q'}^-$, $e_q^+ = e_{q+Q'}^+$, $0 \leq q < Q$, and where two such pairs of cycles are considered as equivalent if one pair is obtained from the other pair by cyclically permuting the pairs of edges that make up the pair of cycles (compare here the second to last paragraph of Section 2 of [HIK]).

2. \mathcal{R} -GRAPH SEMIGROUPS

Let \mathcal{S} be a semigroup (with zero). We set

$$\begin{aligned}\mathcal{C}_-(F) &= \{G \in \mathcal{S} : GF \neq 0\}, \\ \mathcal{C}_+(F) &= \{G \in \mathcal{S} : FG \neq 0\}, \quad F \in \mathcal{S}.\end{aligned}$$

We denote by $\mathcal{U}_{\mathcal{S}}$ the set of idempotents U of \mathcal{S} , such that one has for $F_- \in \mathcal{C}_-(U)$ that $F_-U = F_-$ and for $F_+ \in \mathcal{C}_+(U)$ that $UF_+ = F_+$. We make the assumption:

(A) For $H \in \mathcal{S}$ there exist $V, W \in \mathcal{U}_{\mathcal{S}}$ such that $UHW \neq 0$.

We set

$$\begin{aligned}\mathcal{S}^-(U) &= \bigcap_{F \in \mathcal{C}_-(U)} \mathcal{C}_+(FU), \quad U \in \mathcal{U}_{\mathcal{S}}, \\ \mathcal{S}^-(V, W) &= \mathcal{S}^-(V)W \setminus \{0\}, \quad V, W \in \mathcal{U}_{\mathcal{S}},\end{aligned}$$

and, symmetrically, we set

$$\begin{aligned}\mathcal{S}^+(U) &= \bigcap_{F \in \mathcal{C}_+(U)} \mathcal{C}_-(UF), \quad U \in \mathcal{U}_{\mathcal{S}}, \\ \mathcal{S}^+(V, W) &= W\mathcal{S}^+(V) \setminus \{0\}, \quad W, V \in \mathcal{U}_{\mathcal{S}}.\end{aligned}$$

By (A) we can define sub-semigroups (with zero) \mathcal{S}^- and \mathcal{S}^+ of \mathcal{S} by setting

$$\mathcal{S}^- = \bigcup_{U \in \mathcal{U}_{\mathcal{S}}} \mathcal{S}^-(U), \mathcal{S}^+ = \bigcap_{F \in \mathcal{C}_+(U)} \mathcal{C}_-(UF), \quad U \in \mathcal{U}_{\mathcal{S}},$$

We formulate assumptions (AP 1– 3) and (AQ 1 – 2) on the semigroup \mathcal{S} :

(AP1) For $U, V \in \mathcal{U}_{\mathcal{S}}$, if $UV \neq 0$, then $U = V$.

(AP2) $\mathcal{U}_{\mathcal{S}}$ is a finite set.

(AP3) $\mathcal{C}_+(V) \cap \mathcal{C}_-(W) \neq \emptyset$, $V, W \in \mathcal{U}_{\mathcal{S}}$.

(AQ1) $\mathcal{S}^-(V, U)\mathcal{S}^+(W, U) \subset \mathcal{S}^-(V)\mathcal{S}^+(W)$, $U, V, W \in \mathcal{U}_{\mathcal{S}}$.

(AQ2) $\mathcal{S} = \bigcup_{U \in \mathcal{U}_{\mathcal{S}}} \mathcal{S}^+(U)\mathcal{S}^-(U)$.

We also have conditions (AQ 3 – 6) each of which comes in two parts that are symmetric to one another. We only write one part of these conditions.

(AQ3–) For $U, V \in \mathcal{U}_{\mathcal{S}}$ one has that for $F^- \in \mathcal{S}^-(U, V)$ there exists an $F^+ \in \mathcal{S}^+(V, U)$ such that

$$F^-F^+ = U.$$

(AQ4–) For $U, V, W \in \mathcal{U}_{\mathcal{S}}$ one has that for

$$F^- \in \mathcal{S}^-(U, V) \setminus \{U\}, G^+ \in \mathcal{S}^+(V, W) \setminus \{V\}$$

such that

$$F^-G^+ \in \mathcal{S}^-(U, W),$$

there exists an $H^- \in \mathcal{S}^-(W, V) \setminus \{W\}$ such that

$$F^-G^+H^- = F.$$

We say that an element $F \in \mathcal{S}^- \setminus \mathcal{U}_{\mathcal{S}}$ is indecomposable in \mathcal{S}^- if $F = GH, G, H \in \mathcal{S}^-$, implies that G or H is in $\mathcal{U}_{\mathcal{S}}$. The indecomposable elements in \mathcal{S}^+ are defined symmetrically.

(AQ5–) \mathcal{S}^- has finitely many indecomposable elements.

(AQ6–) \mathcal{S}^- is generated by its indecomposable elements.

For an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we set

$$\begin{aligned}\Omega^+(e^-) &= \{e^+ \in \mathcal{E}^+(\mathfrak{r}, \mathfrak{q}) : (e^-, e^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r})\}, & e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), \\ \Omega^-(e^+) &= \{e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) : (e^-, e^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r})\}, & e^+ \in \mathcal{E}^+(\mathfrak{r}, \mathfrak{q}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},\end{aligned}$$

and we introduce a Condition (a) on the \mathcal{R} -graph, that consists of two parts (a –) and (a +), that are symmetric to one another

$$\begin{aligned}(\text{a } -) \quad & \Omega^+(e^-) \neq \Omega^+(\tilde{e}^-), & e^-, \tilde{e}^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), e^- \neq \tilde{e}^-, & \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}. \\ (\text{a } +) \quad & \Omega^-(e^+) \neq \Omega^-(\tilde{e}^+), & e^+, \tilde{e}^+ \in \mathcal{E}^+(\mathfrak{r}, \mathfrak{q}), e^+ \neq \tilde{e}^+, & \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.\end{aligned}$$

For the proof of the theorem compare [?].

Theorem 2.1. *Let the semigroup \mathcal{S} be such that the projection of \mathcal{S} onto $[\mathcal{S}]$ is an isomorphism. Then \mathcal{S} is an \mathcal{R} -graph semigroup if and only if \mathcal{S} satisfies (AP 1 – 3) and (AQ 1 – 6).*

Proof. We prove sufficiency. We use a partitioned graph with vertex set $\mathcal{U}_{\mathcal{S}}$, and edge sets

$$\mathcal{E}^- = \bigcup_{V, W \in \mathcal{U}_{\mathcal{S}}} \mathcal{E}^-(V, W), \quad \mathcal{E}^+ = \bigcup_{V, W \in \mathcal{U}_{\mathcal{S}}} \mathcal{E}^+(V, W),$$

with the set of indecomposable elements in $\mathcal{S}^-(V, W)$ as the set $\mathcal{E}^-(V, W)$, and the set of indecomposable elements in $\mathcal{S}^+(V, W)$ as the set $\mathcal{E}^+(V, W)$, $V, W \in \mathcal{U}_{\mathcal{S}}$.

Let $U, W \in \mathcal{U}_{\mathcal{S}}$, and let $E^- \in \mathcal{S}^-(V, W)$ be indecomposable in \mathcal{S}^- . By (AQ3–) there exists there exists an $E^+ \in \mathcal{S}^+(V, W)$ such that

$$(1) \quad E^- E^+ = V.$$

We prove that E^+ is indecomposable. Assume the contrary, and let $U \in \mathcal{U}_{\mathcal{S}}$ and

$$F^+ \in \mathcal{S}^+(U, W), \quad G^+ \in \mathcal{S}^+(V, U),$$

be such that

$$(2) \quad E^+ = F^+ G^+, \quad G^+ \neq V.$$

By (AQ2) then either

$$E^- F^+ \in \mathcal{S}^-(V) \setminus \{V\},$$

in which case there would exist by (AQ5) an $F^- \in \mathcal{S}^-(U, W) \setminus \{U\}$, such that

$$E^- = E^- F^+ F^-,$$

contradicting the indecomposability of E^- , or

$$E^- F^+ \in \mathcal{S}^+(W),$$

in which case

$$E^- F^+ G^+ \in \mathcal{S}^+(V) \setminus \{V\},$$

contradicting (1) and (2). The symmetric argument shows also for $U, W \in \mathcal{U}_{\mathcal{S}}$, and for an $E^+ \in \mathcal{S}^+(U, W)$, that is indecomposable in \mathcal{S}^+ , that there exists an $E^- \in \mathcal{S}^-(U, W)$, that is indecomposable in \mathcal{S}^- , such that

$$E^- E^+ = U.$$

It follows that the partitioned graph $(\mathcal{U}_{\mathcal{S}}, \mathcal{E}^-, \mathcal{E}^+)$ is strongly connected.

We define relations $\mathcal{R}(V, W) \subset \mathcal{E}^-(V, W) \times \mathcal{E}^+(V, W)$ by

$$\mathcal{R}(V, W) = \{(E^-, E^+) \in \mathcal{E}^-(V, W) \times \mathcal{E}^+(V, W) : E^- E^+ \neq 0\}, \quad V, W \in \mathcal{U}_S,$$

and set

$$\mathcal{R} = \bigcup_{V, W \in \mathcal{U}_S} \mathcal{R}(V, W).$$

By (AP1) and (AQ4) one has

$$E^- E^+ = \begin{cases} U, & \text{if } U = W, E^- \sim \mathcal{R}(U, V) E^+, \\ 0, & \text{if } U = W, E^- \not\sim \mathcal{R}(U, V) E^+, \\ 0, & \text{if } U \neq W, \end{cases}$$

$$E^- \in \mathcal{E}^-(U, V), E^+ \in \mathcal{E}^-(W, V), \quad U, V, W \in \mathcal{U}_S.$$

It follows from (AQ2), (AQ5), and (AQ6) that for $F \in \mathcal{S}$ there exist $I(-), I(+) \in \mathbb{Z}_+$ and

$$U_{i_+}(+) \in \mathcal{U}_S, \quad I_+ \geq i_+ \geq 1, \quad U \in \mathcal{U}, \quad U_{i_-}(-) \in \mathcal{U}_S, \quad 1 \leq i_- \leq I_-,$$

and, setting

$$U_0(+) = U_0(-) = U,$$

also

$$E_{i_+}^+ \in \mathcal{E}^+(U_{i_+}, U_{i_+-1}), \quad I_+ \geq i_+ \geq 1, \quad E_{i_-}^- \in \mathcal{E}^+(U_{i_- -1}, U_{i_-}), \quad 1 \leq i_- \leq I_-,$$

such that

$$(3) \quad F = \left(\prod_{I_+ \geq i_+ \geq 1} E_{i_+}^+ \right) U \left(\prod_{1 \leq i_- \leq I_-} E_{i_-}^- \right).$$

It follows from the assumption that the projection of \mathcal{S} onto $[\mathcal{S}]$ is an isomorphism, that the \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathcal{U}_S, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Condition (a), since for $E, \tilde{E} \in \mathcal{E}^-(U, V)$ one has that $\Omega^+(E) = \Omega^+(\tilde{E})$ would imply that $[E] = [\tilde{E}], U, V \in \mathcal{U}_S$. Applying Conditions (a-) and (a+) repeatedly, and keeping in mind that one has for $H \in \mathcal{S}, U \in \mathcal{U}_S$ that $[H] = [U]$ implies that $H = U$, one finds that the presentation (3) of F is in fact unique. \square

By a similar argument we characterize the \mathcal{R} -graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ whose projection onto $[\mathcal{S}]$ is an isomorphism.

Theorem 2.2. *The projection of an \mathcal{R} -graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ onto $[\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)]$ is an isomorphism if and only if the \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies Condition (a), and if*

$$\text{card}(\mathcal{E}^-(\eta(p), p)) > 1, \quad p \in \mathfrak{P}^{(1)}, \quad (d-)$$

or, equivalently, if

$$\text{card}(\mathcal{E}^+(\eta(P), P)) > 1, \quad p \in \mathfrak{P}^{(1)}. \quad (d+)$$

Proof. For the proof of necessity we note that if for a $\mathfrak{p} \in \mathfrak{P}^{(1)}$ such that

$$\text{card}(\mathcal{E}^-(\eta(P), P)) = 1,$$

as a consequence of Condition (a) also $\text{card}(\mathcal{E}^-(\eta(P), P)) = 1$, and then with $\mathcal{E}^+(\eta(P), P) = \{e^+\}, \mathcal{E}^-(\eta(P), P) = \{e^-\}$, one has $[e^+e^-] = [\mathbf{1}_{\mathfrak{p}}]$.

To proof sufficiency one shows that Conditions (a) and (d) imply that elements F of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that have identical images in $[\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)]$ also have identical presentations

$$F = \left(\prod_{I_+ \geq i_+ \geq 1} e_{i_+}^+ \right) \mathbf{1}_{\mathfrak{p}} \left(\prod_{1 \leq i_- \leq I_-} e_{i_-}^- \right).$$

By means of Conditions $(a-), (a+)$ and $(d-), (d+)$ this can be reduced to the case that

$$[\mathbf{1}_q(\prod_{I_+ \geq i_+ \geq 1} e_{i_+}^+) \mathbf{1}_p(\prod_{1 \leq i_- \leq I_-} e_{i_-}^-) \mathbf{1}_q] = [\mathbf{1}_q],$$

and in this case it follows also from Conditions $(a-), (a+)$ and $(d-), (d+)$ that $I_+ = I_- = 0, \mathbf{1}_q = \mathbf{1}_p$. \square

3. SUBSHIFTS WITH PROPERTY (A) TO WHICH \mathcal{R} -GRAPH SEMIGROUPS ARE ASSOCIATED

We introduce terminology and notation for subshifts. Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, i, k \in \mathbb{Z}, i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}, \quad i, k \in \mathbb{Z}, i \leq k.$$

We use similar notation also for blocks,

$$b_{[i',k']} = (b_j)_{i' \leq j \leq k'}, \quad b \in X_{[i,k]}, \quad i \leq i' \leq k' \leq k,$$

and also if indices range in semi-infinite intervals. The symbol that denotes a block is also used to denote the word that is carried by the block. We identify the elements of $X_{(-\infty,0)}$ with the left-infinite words that they carry, and we identify the elements of $X_{(0,\infty)}$ with the right-infinite words that they carry. For the higher block system of a subshift $X \subset \Sigma^{\mathbb{Z}}$ we use the notation

$$\begin{aligned} x^{\langle [m,n] \rangle} &= (x_{[i+m, i+n]})_{i \in \mathbb{Z}}, \quad x \in X, \\ X^{\langle [m,n] \rangle} &= \{x^{\langle [m,n] \rangle} : x \in X\}, \quad m, n \in \mathbb{Z}, m < n. \end{aligned}$$

We set

$$\begin{aligned} \Gamma(a) &= \{(x^-, x^+) \in X_{(-\infty,0)} \times X_{(0,\infty)} : (x^-, a, x^+) \in X\}, \\ \Gamma_n^+(a) &= \{b \in X_{(k, k+n]} : (a, b) \in X_{[i, k+n]}\}, \quad n \in \mathbb{N}, \\ \Gamma_\infty^+(a) &= \{y^+ \in X_{(k,\infty)} : (a, y^+) \in X_{[i,\infty)}\}, \\ \Gamma^+(a) &= \Gamma_\infty^+(a) \cup \bigcup_{n \in \mathbb{N}} \Gamma_n^+(a), \quad a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k. \end{aligned}$$

Γ^- has the time symmetric meaning. We set

$$\begin{aligned} \omega_n^+(a) &= \bigcap_{x^- \in \Gamma_\infty^-(a)} \{b \in X_{(k, k+n]} : (x^-, a, b) \in X_{(-\infty, k+n]}\}, \\ \omega_\infty^+(a) &= \bigcap_{x^- \in \Gamma_\infty^-(a)} \{y^+ \in X_{(k,\infty)} : (x^-, a, y^+) \in X\}, \\ \omega^+(a) &= \omega_\infty^+(a) \cup \bigcup_{n \in \mathbb{N}} \omega_n^+(a), \quad a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k. \end{aligned}$$

ω^- has the time symmetric meaning.

We recall that, given subshifts $X \subset \Sigma^{\mathbb{Z}}, \bar{X} \subset \bar{\Sigma}^{\mathbb{Z}}$, and a topological conjugacy $\varphi : X \rightarrow \bar{X}$, there is for some $L \in \mathbb{Z}_+$ a block mapping

$$\Phi : X_{[-L,L]} \rightarrow \bar{\Sigma}$$

such that

$$\varphi(x) = (\Phi(x_{[i-L, i+L]}))_{i \in \mathbb{Z}}.$$

We say then that φ is given by Φ , and we write

$$\Phi(a) = (\Phi(a_{[j-L, j+L]}))_{i+L \leq j \leq k-L}, \quad a \in X_{[i,k]}, \quad i, k \in \mathbb{Z}, k-i \geq 2L,$$

and use similar notation if indices range in semi-infinite intervals.

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ set

$$A_n^-(X) = \bigcap_{i \in \mathbb{Z}} \{x \in X : x_i \in \omega^+(x_{[i-n, i)})\}, \quad n \in \mathbb{N},$$

and

$$A^-(X) = \bigcup_{n \in \mathbb{N}} A_n^-(X).$$

Define $A_n^+(X)$, $n \in \mathbb{N}$, and $A^+(X)$ symmetrically and set

$$A_n(X) = A_n^-(X) \cap A_n^+(X), \quad n \in \mathbb{N},$$

and

$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

Denote the set of periodic points in $A(X)$ by $P(A(X))$. We write for $q, r \in P(A(X))$, $q \succeq r$, if there exists a point in $A(X)$ that is left asymptotic to the orbit of q and right asymptotic to the orbit of r . The equivalence relation that results from the preorder relation \succeq we write \approx and we denote the order relation that results from \approx also by \succeq . We denote the set of \approx -equivalence classes by $\mathfrak{P}(X)$. The order structure $(\mathfrak{P}(X), \approx)$ is invariantly associated to X [Kr2].

We write a condition (DP0) on a subshift $X \subset \Sigma^{\mathbb{Z}}$, that is invariant under topological conjugacy [Kr2]:

(DP0) $P(A(X))$ is dense in X .

The order structure $(\mathfrak{P}(X), \approx)$ being invariantly associated to X , the following conditions (DP1) and (DP2) are invariant under topological conjugacy. Condition (DP1) is the translation of Condition (AP1) and Condition (DP2) is the translation of Condition (AP2):

(DP1) $\mathfrak{P}(X)$ is a finite set.

(DP2) For $q, r \in P(A(X))$, if $q \succeq r$ then $q \approx r$.

We introduce a Condition (DQ1) and a Condition (DQ2). Condition (DQ1) is the translation of Condition (AQ1) and Condition (DQ2) is the translation of Condition (AQ2):

(DQ1) For $p \in P(A(X))$, there exists an $H \in \mathbb{N}$, such that, if

$$\begin{aligned} y &\in A^-(X) \cap Y(X), & y_{(-H, \infty)} &= p_{(-H, \infty)}, \\ z &\in A^+(X) \cap Y(X), & z_{(-\infty, H]} &= p_{(-\infty, H]}, \end{aligned}$$

and

$$(y_{(-\infty, 0]}, z_{(0, \infty)}) \in X,$$

then

$$(y_{(-\infty, 0]}, z_{(0, \infty)}) \in A^-(X) \cup A^+(X).$$

(DQ2) Given $q, r \in P(A(X))$, there exists an $H \in \mathbb{N}$, such that the following holds: For $x \in X$ and $K > H$, such that

$$x_{(-\infty, -K]} = q_{(-\infty, 0]}, \quad x_{(K, \infty)} = r_{(0, \infty)},$$

and for $M \in \mathbb{N}$, there exist $p \in P(A(X))$, $I, J > M$, and

$$y \in A^+(X) \cap Y(X), \quad z \in A^-(X) \cap Y(X),$$

such that

$$y_{(-\infty, -I]} = q_{(-\infty, 0]}, \quad y_{(M, \infty)} = p_{(M, \infty)},$$

$$z_{(-\infty, M]} = p_{(-\infty, M]}, \quad z_{(J, \infty)} = r_{(0, \infty)},$$

and such that

$$\Gamma(x_{(-K-M, K+M]}) = \Gamma((y_{(-I-H, 0]}, z_{(0, J+H]})).$$

Next we introduce a Condition (DQ3) and a Condition (DQ4). Condition (DQ3) comprises two conditions that we name (DQ3-) and (DQ3+), and that are symmetric to one another. Also Condition (DQ4) comprises two conditions that we name (DQ4-) and (DQ4+), and that are symmetric to one another. We write only Condition (DQ3-) and Condition (DQ4-). Condition (DQ3-) is the translation of Condition (AQ3-) and Condition (DQ4-) is the translation of Condition (AQ4-):

(DQ3-) For $p \in P(A(X))$ and

$$x \in A^-(X) \cap Y(X), \quad x_{(0, \infty)} = p_{(0, \infty)},$$

and for $M \in \mathbb{N}$ there exists a $y \in A^+(X) \cap Y(X)$, that is right asymptotic to the orbit of the periodic point to which x is left asymptotic, such that

$$y_{(-\infty, M]} = p_{(-\infty, M]},$$

and such that

$$(x_{(-\infty, 0]}, y_{(0, \infty)}) \in A(X).$$

(DQ4-) For $q, r \in P(A(X))$ there exist an $H \in \mathbb{N}$ such that for $K > H$ and for $x \in A^-(X), y \in A^+(X)$ such that

$$x_{(0, \infty)} = q_{(0, \infty)}, \quad y_{(-\infty, K]} = q_{(-\infty, K]},$$

$$(x_{(-\infty, K]}, q_{(0, \infty)}) \in A^-(X),$$

there exist $z \in A^-(X)$ and $I, J \in \mathbb{N}, K < I < J$, such that

$$z_{(-\infty, I+K]} = r_{(-\infty, I+K]}, \quad z_{(J, \infty)} = q_{(0, \infty)},$$

$$(x_{(-\infty, K]}, y_{(K, I]}, z_{(I, \infty)}) \in A^-(X),$$

and

$$\Gamma^+(x_{(-\infty, K]}) = \Gamma^+(x_{(-\infty, K]}, y_{(K, J+K]}, z_{(I, J+K]}).$$

We call a point $y \in A^+(X)$ indecomposable, if there is an $H \in \mathbb{N}$, such that, with p the point in $P(A(X))$ to which y is right asymptotic, the following holds for $K > H$: For $I \in \mathbb{Z}$ such that

$$x_{(I, \infty)} = p_{(I, \infty)},$$

and for

$$y \in Y(X), \quad J \in \mathbb{Z},$$

such that

$$y_{(-\infty, J+K]} = p_{(-\infty, J+K]},$$

if

$$(x_{(-\infty, I]}, y_{(J, \infty)}) \in X,$$

then

$$(x_{(-\infty, I]}, y_{(J, \infty)}) \in A^+(X).$$

With this notion of indecomposable point we translate Condition (AQ5-) into a Condition (DQ5-), that together with its symmetric counter part (DQ5+) is Condition (DQ5):

(DQ5-) For $q, r \in P(A(X))$ there exist $H, M \in \mathbb{N}$ such that for an indecomposable point $x \in A^+(X) \cap Y(X)$ and for $I_-, I_+ \in \mathbb{Z}, I_- < I_+$, such that

$$x_{(-\infty, I_-]} = q_{(-\infty, 0]}, \quad x_{(I_+, \infty)} = r_{(0, \infty)},$$

there exists an indecomposable point $y \in A^+(X) \cap Y(X)$ and $J_-, J_+ \in \mathbb{Z}$, $J_- < J_+$, such that

$$J_+ - J_- \leq M,$$

$$y_{(-\infty, J_-]} = q_{(-\infty, 0]}, \quad y_{(J_+, \infty)} = r_{(0, \infty)},$$

and

$$\Gamma^+(x_{(I_-, H, I_+ + H]}) = \Gamma^+(y_{(J_-, H, J_+ + H]}).$$

We have a Condition (DQ6) that also consists of two parts (DQ6−) and (DQ6+), that are symmetric to one another:

(DQ6−) If $x^- \in X_{(-\infty, 0]}$ and $I_k \in \mathbb{N}$, $I_k > I_k, k \in \mathbb{N}$, such that

$$x_{-I_k}^- \notin \omega_1^-(x_{(-I_k, 0]}^-), \quad k \in \mathbb{N},$$

then for $y_{(-\infty, J]}^- \in X_{(-\infty, J]}$, $J \in \mathbb{N}$, such that

$$y_{(-\infty, 0]}^- = x_{(-\infty, 0]}^-,$$

there is a k_o such that

$$y_{-I_k}^- \notin \omega_1^-(y_{(-I_k, 0]}^-), \quad k \geq k_o.$$

Condition (DQ6−) appeared in connection with the Cantor horizon of the Dyck shift [KM2]. Inspection shows that conditions (DQ1−) and (DQ2−), and therefore also conditions (DQ1+) and (DQ2+), are invariant under topological conjugacy. Also, a topological conjugacy maps indecomposable points to indecomposable points. As a consequence, Condition (DQ5−), and therefore also Condition (DQ5+), is invariant under topological conjugacy. We prove that Condition (DQ6−) is invariant under topological conjugacy.

Proposition 3.1. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$, be subshifts and let $\varphi : X \rightarrow \tilde{X}$ be a topological conjugacy. Let X satisfy Condition (DQ6−). Then \tilde{X} also satisfies Condition (DQ6−).*

Proof. let $L \in \mathbb{Z}_+$ be such that $[-L, L]$ is a coding window for φ and φ^{-1} , and let φ^{-1} be given by the block map $\Phi : \tilde{X}_{[-L, L]} \rightarrow \Sigma$.

Let $\tilde{x}^- \in \tilde{X}_{(-\infty, 0]}$, and let $I_k \in \mathbb{N}$,

$$I_{k+1} > I_k, \quad k \in \mathbb{N},$$

be such that

$$\tilde{x}_{-I_k}^- \notin \omega_1^-(\tilde{x}_{(-I_k, 0]}^-), \quad k \in \mathbb{N}.$$

Then for

$$x^- = \Phi(\tilde{x}^-),$$

and for $k \in \mathbb{N}$ such that $I_k > 2L$ there exists an $i \in \mathbb{Z}$,

$$I_k - L < i \leq I_k + L,$$

such that

$$x_i^- \notin \omega_1^-(x_{(i, -L]}^-).$$

Let

$$J \in \mathbb{N}, \quad \tilde{y}^- \in \tilde{X}_{(-\infty, J]},$$

such that

$$\tilde{y}_{(\infty, 0]}^- = \tilde{x}^-,$$

and set

$$y^- = \Phi(\tilde{y}^-).$$

Also let $M \in \mathbb{N}$. From Condition (DQ6-) for X it follows that there is an $I \in \mathbb{N}$,

$$(1) \quad I > M + L,$$

such that

$$y_{-I}^- \notin \omega_1^-(y_{(-I, J-L]}^-).$$

This implies that

$$\tilde{y}_{(-I-L, I+L]}^- \notin \omega_{2L+1}^-(\tilde{y}_{(-J-L, J]}^-).$$

It follows that there exists an $\tilde{I} \in \mathbb{Z}$,

$$(2) \quad -I - L < \tilde{I} \leq -I + L,$$

such that

$$(3) \quad \tilde{y}_{(-I, \tilde{I}]}^- \notin \omega_{\tilde{I}-I}^-(\tilde{y}_{(-\tilde{I}, J]}^-).$$

By (1), (2) and (3) Condition (DQ6-) is satisfied by \tilde{X} . \square

We recall from [?] the definition of property (A). For $n \in \mathbb{N}$ a subshift $X \subset \Sigma^{\mathbb{Z}}$ such that $A_1(Y) \neq \emptyset$, has property (a, n, H) , $H \in \mathbb{N}$, if for $h, \tilde{h} \geq 3H$ and for

$$a \in A_n(X)_{[1, h]}, \quad \tilde{a} \in A_n(X)_{[1, \tilde{h}]},$$

such that

$$a_{[1, H]} = \tilde{a}_{[1, H]}, \quad a_{(h-H, h]} = \tilde{a}_{(\tilde{h}-H, \tilde{h}]},$$

one has that a and \tilde{a} have the same context. A subshift $X \subset \Sigma^{\mathbb{Z}}$ has Property (A) if there are $H_n, n \in \mathbb{N}$, such that X has the properties (a, n, H_n) , $n \in \mathbb{N}$.

We also recall the construction of the associated semigroup. For a property (A) subshift $X \subset \Sigma^{\mathbb{Z}}$ we denote by Y_X the set of points in X that are left asymptotic to a point in $P(A(X))$ and also right-asymptotic to a point in $P(A(X))$. Let $y, \tilde{y} \in Y_X$, let y be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let \tilde{y} be left asymptotic to $\tilde{q} \in P(A(X))$ and right asymptotic to $\tilde{r} \in P(A(X))$. Given that X has the properties (a, n, H_n) , $n \in \mathbb{N}$, we say that y and \tilde{y} are equivalent, $y \approx \tilde{y}$, if $q \approx \tilde{q}$ and $r \approx \tilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \tilde{q}, \tilde{r} \in X_n(Y)$ and for $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}$, $I < J, \tilde{I} < \tilde{J}$, such that

$$\begin{aligned} y_{(-\infty, I]} &= q_{(-\infty, 0]}, & y_{(J, \infty)} &= r_{(0, \infty)}, \\ \tilde{y}_{(-\infty, \tilde{I}]} &= \tilde{q}_{(-\infty, 0]}, & \tilde{y}_{(\tilde{J}, \infty)} &= \tilde{r}_{(0, \infty)}, \end{aligned}$$

one has for $h \geq 3H_n$ and for

$$a \in Y_{(I-h, J+h]}, \quad \tilde{a} \in Y_{(\tilde{I}-h, \tilde{J}+h]},$$

such that

$$\begin{aligned} a_{(I-H_n, J+H_n]} &= y_{(I-H_n, J+H_n]}, & \tilde{a}_{(\tilde{I}-H_n, \tilde{J}+H_n]} &= \tilde{y}_{(\tilde{I}-H_n, \tilde{J}+H_n]}, \\ a_{(I-h, I-h+H_n)} &= \tilde{a}_{(\tilde{I}-h, \tilde{I}-h+H_n)}, \\ a_{(J+h-H_n, J+h]} &= \tilde{a}_{(\tilde{J}+h-H_n, \tilde{J}+h]}, \end{aligned}$$

and such that

$$\begin{aligned} a_{(I-h, I]} &\in A_n(X)_{(I-h, I]}, & \tilde{a}_{(\tilde{J}-h, \tilde{I}]} &\in A_n(X)_{(\tilde{J}-h, \tilde{I}]}, \\ a_{(J, J+h]} &\in A_n(X)_{(J, J+h]}, & \tilde{a}_{(\tilde{J}, \tilde{J}+h]} &\in A_n(X)_{(\tilde{J}, \tilde{J}+h]}, \end{aligned}$$

that a and \tilde{a} have the same context. To give $[Y_X]_{\approx}$ the structure of a semigroup $\mathcal{S}(X)$, that is invariantly associated to X , let $u, v \in Y_X$, let u be right asymptotic to $q \in P(A(X))$ and let v be left asymptotic to $r \in P(A(X))$. If here $q \succ r$, then $[u]_{\approx}[v]_{\approx}$ is set equal to $[y]_{\approx}$ where y is any point in Y such that there are $n \in \mathbb{N}$, $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}$, $I < J, \tilde{I} < \tilde{J}$, such that $q, r \in A_n(X)$, and such that

$$u_{(I, \infty)} = q_{(I, \infty)}, \quad v_{(-\infty, J]} = r_{(-\infty, J]},$$

$$y_{(-\infty, \hat{I}+H_n]} = u_{(-\infty, I+H_n]}, \quad y_{(\hat{J}-H_n, \infty)} = v_{(J-H_n, \infty)},$$

and

$$y_{(\hat{I}, \hat{J}]} \in A_n(X)_{(\hat{I}, \hat{J}]},$$

provided that such a point y exists. If such a point y does not exist, $[u]_{\approx}[v]_{\approx}$ is set equal to zero. Also, in the case that $q \neq r$, $[u]_{\approx}[v]_{\approx}$ is set equal to zero.

For a subshift with Property (A) Conditions (AP1), (AP2), (AQ1), (AQ 2) and (AQ4) are equivalent to their translations. We prove that for a subshift with Property (A) Condition (DQ4) and Condition (DQ6) together imply Condition (AQ6).

Proposition 3.2. *Let the subshift X have property (A) and let Condition (DQ4) and Condition (DQ6) be satisfied by X . Then $\mathcal{S}(X)$ satisfies Condition (AQ6).*

Proof. If $\mathcal{S}(X)$ does not satisfy Condition (AQ6) then there exist $H_0^- \in \mathcal{S}^-(X)$ and

$$H_m^-, G_m^- \in \mathcal{S}^-(X), \quad m \in \mathbb{N},$$

such that

$$H_m^- = H_{m-1}^- G_m^-, \quad m \in \mathbb{N},$$

or there exist $H_0 \in \mathcal{S}^-(X)$ and

$$H_m^-, G_m^- \in \mathcal{S}^-(X), \quad m \in \mathbb{N},$$

such that

$$H_m^- = G_m^- H_{m-1}^-, \quad m \in \mathbb{N}.$$

Assume the first case. Let $U \in \mathcal{U}_S$ and $U_m \in \mathcal{U}_S, m \in \mathbb{Z}_+$, be given by

$$U H_0^- U_0 \neq 0,$$

$$H_m G_m^- U_{m-1} \neq 0, \quad m \in \mathbb{N},$$

and let

$$G_m^+ \in \mathcal{S}^+(X)(U_m, U_{m-1}), \quad \tilde{G}_m^+ \in \mathcal{S}^+(X)(U_{m-1}, U_m),$$

be such that

$$G_m^- G_m^+ = U_m, \quad G_m^- \tilde{G}_m^+ = 0, \quad m \in \mathbb{N}.$$

Also let $p \in P(A(X))$ and $p^{(m)} \in P(A(X)), m \in \mathbb{N}$, be such that

$$[p]_{\sim} = U,$$

$$[p^{(m)}]_{\sim} = U_m, \quad m \in \mathbb{Z}_+$$

With an appropriately choosen h_o , let $x^- \in X_{(-\infty, 0]}$ and $J_m^- \in \mathbb{N}, J_0^- = 0$,

$$J_m^- \geq J_{m-1}^- + h_o,$$

such that

$$x_{(-J_m^-, -J_m^- + h_o]}^- = p_{(0, h_o]}^{(m)},$$

$$[(p_{(-\infty, h_o]}^{(m)}, x_{(-J_m^- + h_o, -J_{m-1}^- \circ,]} p_{(0, \infty)}^{(m-1)})]_{\sim} = G_m^-, \quad m \in \mathbb{N},$$

and

$$J_m^+, \tilde{J}_m^+ \in \mathbb{N}, \quad J_0^+ = 0,$$

$$a^{(M)} \in X_{(0, \tilde{J}_M^+]}, \quad M \in \mathbb{N},$$

such that

$$a_{(J^+, J^+ + h_o]}^{(M)} = p_{(0, h_o]}^{(m)}, \quad 1 \leq m \leq M,$$

$$[(p_{(-\infty, h_o]}^{(m-1)}, a_{(J^+, J^+ + h_o]}^{(M)}, p_{(0, \infty)}^{(m)})]_{\sim} = G_m^+, \quad 1 \leq m \leq M,$$

$$[(p_{(-\infty, h_o]}^{(M-1)}, a_{(J_M^+, \tilde{J}_M^+]}^{(M)}, p_{(0, \infty)}^{(M)})]_{\sim} = \tilde{G}_M^+, \quad M \in \mathbb{N}.$$

Then

$$(x_{(-J_M^-, 0]}^-, a^{(M)}) \in X_{(-J_M^-, J_M^+]}, \quad (x_{(-J_{M+1}^-, 0]}^-, a^{(M)}) \notin X_{(-J_{M+1}^-, J_{M+1}^+]}.$$

It follows that there is an $i \in \mathbb{Z}$, $-J_M < i \leq -J_{M-1}$, such that $x_i^- \notin \omega_1^-(x_{(i, 0]}^-)$. However, with $J \in \mathbb{N}$, $b \in X_{(0, J]}$, such that

$$b_{(0, h_o]} = p_{(0, h_o]}, \\ [(p_{(-\infty, h_o]}^{(0)}, b_{(h_o, J]}, p_{(0, \infty)})] \sim = H^+,$$

one has

$$(x_{(-J_m^-, 0]}^-, b) \in X_{(-\infty, J]}, \quad m \in \mathbb{N},$$

and therefore

$$(x^-, b) \in X_{(-\infty, J]},$$

contradicting condition (QA-).

The second case reduces by Condition (DQ4+) to the first case. \square

Corollary 3.3. *A subshift with property (A) that satisfies conditions (DP1-2) and (DQ1-6) has as its associated semigroup an \mathcal{R} -graph semigroup.*

Proof. Apply Theorem (2.1) and Proposition (3.2), taking into account that conditions (DP1) and (DP2) are the translations of conditions (AP1) and (AP2) and that conditions (DQ3), (DQ4) and (D5) are the translations of conditions (AQ3), (AQ4) and (AQ5). \square

4. PROPERTY (B)

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (B) with respect to the parameter $M \in \mathbb{N}$ if the following holds: For $I_-, I_+, J_-, J_+ \in \mathbb{Z}$ such that

$$I_+ - I_-, J_+ - J_- \geq M,$$

and

$$a \in X_{(I_-, I_+]}, \quad b \in X_{(J_-, J_+]},$$

such that

$$a_{(I_-, I_- + M]} = b_{(J_-, J_- + M]}, \quad a_{(I_+ - M, I_+]} = b_{(J_+ - M, J_+]},$$

and $R \in \mathbb{N}$ and

$$x^- \in \omega^-(a) \cap \omega^-(b), \quad x^+ \in \omega^+(a) \cap \omega^+(b),$$

such that

$$\Gamma(x_{(I_- - R, I_-]}^-, a, x_{(I_+, I_+ + R]}^+) = \Gamma(x_{(J_- - R, J_-]}^-, b, x_{(J_+, J_+ + R]}^+),$$

one has that

$$\Gamma(a) = \Gamma(b).$$

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (B) if it has property (B) with respect to some parameter.

Theorem 4.1. *Property (B) is an invariant of topological conjugacy.*

Proof. If a subshift $X \subset \Sigma^{\mathbb{Z}}$ has Property B with respect to some parameter, then its higher block systems also have Property B. To prove the proposition it is therefore enough to consider the situation that one is given subshifts $X \subset \Sigma^{\mathbb{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \rightarrow \tilde{X}$ that is given by a 1-block map $\Phi : \Sigma \rightarrow \tilde{\Sigma}$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map $\tilde{\Phi} : \tilde{X}_{[-L, L]} \rightarrow \Sigma$,

where X has Property (B) with respect to the parameter M , and to prove that the subshift \tilde{X} has also Property (B). We set

$$\widetilde{M} = M + 2L,$$

and we prove that \tilde{X} has Property (B) with respect to the parameter \widetilde{M} . For this let $I_-, I_+, J_-, J_+ \in \mathbb{Z}$,

$$I_+ - I_-, J_+ - J_- \geq \widetilde{M},$$

and let

$$\tilde{a} \in \tilde{X}_{(I_-, I_+]}, \quad \tilde{b} \in \tilde{X}_{(J_-, J_+]},$$

be such that

$$(1) \quad \tilde{a}_{(I_-, I_- + \widetilde{M})} = \tilde{b}_{(J_-, J_- + \widetilde{M})}, \quad \tilde{a}_{(I_+ - \widetilde{M}, I_+]} = \tilde{b}_{(J_+ - \widetilde{M}, J_+]},$$

and let $R \in \mathbb{N}$ and

$$(2) \quad \tilde{x}^- \in \omega^-(\tilde{a}) \cap \omega^-(\tilde{b}), \quad \tilde{x}^+ \in \omega^+(\tilde{a}) \cap \omega^+(\tilde{b}),$$

be such that

$$(3) \quad \Gamma(\tilde{x}_{(I_- - R, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + R]}^+) = \Gamma(\tilde{x}_{(J_- - R, J_-]}^-, \tilde{b}, \tilde{x}_{(J_+, J_+ + R]}^+).$$

We have to prove that

$$(4) \quad \Gamma(\tilde{a}) = \Gamma(\tilde{b}).$$

We let

$$a \in X_{(I_- + L, I_+ - L]}, \quad b \in X_{(J_- + L, J_+ - L]},$$

be given by

$$a = \tilde{\Phi}(\tilde{a}), \quad b = \tilde{\Phi}(\tilde{b}).$$

By (2)

$$(5) \quad a_{(I_- + L, I_- + L + M]} = b_{(J_- + L, J_- + L + M]}, \quad a_{(I_+ - L - M, I_+ - L]} = b_{(J_+ - L - M, J_+ - L]}.$$

We set also

$$x^- = \tilde{\Phi}(\tilde{x}^-, \tilde{a}_{(I_-, I_- + 2L]}), \quad x^+ = \tilde{\Phi}(\tilde{a}_{(I_+ - 2L, I_+]}, \tilde{x}^+).$$

It follows from (1) and (2) that

$$(6) \quad x^- \in \omega^-(a) \cap \omega^-(b), \quad x^+ \in \omega^+(a) \cap \omega^+(b).$$

It is

$$\begin{aligned} \Gamma(x_{(I_- - R - L, I_- + L]}^-, a, x_{(I_+ - L, I_+ + R + L]}^+) = \\ \{(\tilde{\Phi}(\tilde{u}^-), \tilde{\Phi}(\tilde{u}^+)) : (\tilde{u}^-, \tilde{u}^+) \in \Gamma(\tilde{x}_{(I_- - R, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + R]}^+)\} \end{aligned}$$

with a similar expression for $\Gamma(x_{(I_- - R - L, I_- + L]}^-, b, x_{(I_+ - L, I_+ + R + L]}^+)$, and it is seen that (3) implies that

$$(7) \quad \begin{aligned} \Gamma(x_{(I_- - R - L, I_- + L]}^-, a, x_{(I_+ - L, I_+ + R + L]}^+) = \\ \Gamma(x_{(I_- - R - L, I_- + L]}^-, b, x_{(I_+ - L, I_+ + R + L]}^+). \end{aligned}$$

By (5), (6) and (7) we can apply Property (B) of X to obtain

$$(8) \quad \Gamma(a) = \Gamma(b).$$

It is

$$(4.2) \quad \begin{aligned} \Gamma(\tilde{a}) = \{(\Phi(u_{(-\infty, I_-]}^-), \Phi(u_{(I_+, \infty)}^+)) : (u^-, u^+) \in \Gamma(a), \\ \Phi(u_{(I_-, I_+ + L]}^-) = \tilde{a}_{(I_-, I_+ + L]}, \Phi(u_{(I_+, I_+ + L]}^+) = \tilde{a}_{(I_+, I_+ + L]}\} \end{aligned}$$

with a similar expression for $\Gamma(\tilde{b})$, from which it is seen that (8) implies (4). \square

Lemma 4.3. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$, be subshifts, and let $\varphi : X \rightarrow \tilde{X}$ be a topological conjugacy that is given by a 1-block map $\Phi : \Sigma \rightarrow \tilde{\Sigma}$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map $\tilde{\Phi} : \tilde{X}_{[-L, L]} \rightarrow \Sigma$. Let \tilde{X} have property (B) with respect to the parameter L . Let $I_-, I_+, J_-, J_+ \in \mathbb{Z}$ be such that*

$$I_+ - I_-, J_+ - J_- \geq L,$$

and let

$$\tilde{a} \in \tilde{X}_{(I_-, I_+]}, \quad \tilde{b} \in \tilde{X}_{(J_-, J_+]},$$

be such that

$$\tilde{a}_{(I_-, I_+ + L]} = \tilde{b}_{(J_-, J_- + L]}, \quad \tilde{a}_{(I_+ - L, I_+]} = \tilde{b}_{(J_+ - L, J_+]},$$

and let

$$(2) \quad \tilde{x}^- \in \omega^-(\tilde{a}) \cap \omega^-(\tilde{b}), \quad \tilde{x}^+ \in \omega^+(\tilde{a}) \cap \omega^+(\tilde{b}),$$

Set

$$a = \tilde{\Phi}(\tilde{x}_{(I_- - L, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + L]}^+), \quad b = \tilde{\Phi}(\tilde{x}_{(J_- - L, J_-]}^-, \tilde{b}, \tilde{x}_{(J_+, J_+ + L]}^+).$$

Then

$$(1) \quad \Gamma(a) = \Gamma(b),$$

implies

$$(2) \quad \Gamma(\tilde{a}) = \Gamma(\tilde{b}).$$

Proof. It is

$$\begin{aligned} & \Gamma(\tilde{x}_{(I_- - L, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + L]}^+) = \\ & \{(\Phi(u_{(-\infty, I_- - L]}^-), \Phi(u_{(I_+, I_+ + L]}^+)) : (u^-, u^+) \in \Gamma(a), \\ & \Phi(u_{(I_- - L, I_-]}^-) = \tilde{x}_{(I_- - L, I_-]}^-, \Phi(u_{(I_+, I_+ + L]}^+) = \tilde{x}_{(I_+, I_+ + L]}^+\}, \end{aligned}$$

and replacing here in the right hand side a by b yields the corresponding expression for $\Gamma(\tilde{x}_{(J_- - L, J_-]}^-, \tilde{b}, \tilde{x}_{(J_+, J_+ + L]}^+)$. From this it is seen that (1) implies

$$\Gamma(\tilde{x}_{(I_- - L, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + L]}^+) = \Gamma(\tilde{x}_{(I_- - L, I_-]}^-, \tilde{b}, \tilde{x}_{(I_+, I_+ + L]}^+),$$

which then by Property (B) of \tilde{X} implies (2). \square

5. PROPERTY (C)

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (c) with respect to the parameter

$$(P, (Q_n)_{n \in \mathbb{Z}_+}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^{\mathbb{Z}_+},$$

where

$$P + Q_n > n, \quad Q_{n+1} \geq Q_n, \quad n \in \mathbb{Z}_+,$$

if the following holds: For $n \in \mathbb{Z}_+$ and $I_-, I_+, J_-, J_+ \in \mathbb{Z}_+$,

$$I_+ - I_-, J_+ - J_- \geq P + Q_n,$$

and for

$$a \in X_{(I_-, I_+]}, \quad b \in X_{(J_-, J_+]},$$

such that

$$\omega^-(a) \cap \omega^-(b) \neq \emptyset, \quad \omega^+(a) \cap \omega^+(b) \neq \emptyset,$$

and such that

$$\begin{aligned} & a_{(I_-, I_+ + P + Q_n]} = b_{(J_-, J_- + P + Q_n]}, \\ & \Gamma^-(a) \subset \Gamma^-(b_{(J_-, J_+ - n]}), \quad \Gamma^-(b) \subset \Gamma^-(a_{(I_-, I_+ - n]}), \end{aligned}$$

and

$$a_{(I_+ - P - Q_n, I_+]} = b_{(J_+ - P - Q_n, J_+]},$$

$$\Gamma^+(a) \subset \Gamma^+(b_{(J_-+n, J_+]}), \quad \Gamma^+(b) \subset \Gamma^+(a_{(I_-+n, I_+]}),$$

one has that

$$\Gamma(a) = \Gamma(b).$$

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has Property (c) if it has Property (c) with respect to some parameter.

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has Property (C) if it has Property (B) and Property (c).

Proposition 5.1. *Property (C) is an invariant of topological conjugacy.*

Proof. By Proposition 5.3 Property (B) is an invariant of topological conjugacy. Also, if a subshift $X \subset \Sigma^{\mathbb{Z}}$ has Property (c), then its higher block systems also have Property (c). To prove the proposition it is therefore enough to consider the situation that one is given subshifts $X \subset \Sigma^{\mathbb{Z}}$, $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \rightarrow \tilde{X}$ that is given by a 1-block map $\Phi : \Sigma \rightarrow \tilde{\Sigma}$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map $\tilde{\Phi} : \tilde{X}_{[-L, L]} \rightarrow \Sigma$, where X has Property (B) and has Property (c) with respect to the parameter $(P, (Q_n)_{n \in \mathbb{Z}_+})$, and to prove that the subshift $\tilde{X} \subset \tilde{\Sigma}^{\mathbb{Z}}$ has Property (c). We set

$$\tilde{P} = P + L,$$

and

$$\tilde{Q}_n = Q_{n+L}, \quad n \in \mathbb{Z}_+.$$

We prove that \tilde{X} has Property (c) with respect to the parameter $(\tilde{P}, (\tilde{Q}_n)_{n \in \mathbb{Z}_+})$. For this, let $n \in \mathbb{Z}_+$, and $I_-, I_+, J_-, J_+ \in \mathbb{Z}_+$, be such that

$$I_+ - I_-, J_+ - J_- \geq \tilde{P} + \tilde{Q}_n,$$

and let

$$\tilde{a} \in \tilde{X}_{(I_-, I_+]}, \quad \tilde{b} \in \tilde{X}_{(J_-, J_+]},$$

be such that

$$(1) \quad \omega^-(a) \cap \omega^-(b) \neq \emptyset, \quad \omega^+(a) \cap \omega^+(b) \neq \emptyset,$$

and such that

$$(2) \quad \tilde{a}_{(I_-, I_- + \tilde{P} + \tilde{Q}_n]} = \tilde{b}_{(J_-, J_- + \tilde{P} + \tilde{Q}_n]},$$

$$(3) \quad \Gamma^-(\tilde{a}) \subset \Gamma^-(\tilde{b}_{(J_-, J_+ - n]}), \quad \Gamma^-(\tilde{b}) \subset \Gamma^-(\tilde{a}_{(I_-, I_+ - n]}),$$

and

$$\begin{aligned} \tilde{a}_{(I_+ - \tilde{P} - \tilde{Q}_n, I_+]} &= \tilde{b}_{(J_+ - \tilde{P} - \tilde{Q}_n, J_+]}, \\ \Gamma^+(\tilde{a}) &\subset \Gamma^+(\tilde{b}_{(J_-, J_+ + n]}), \quad \Gamma^+(\tilde{b}) \subset \Gamma^+(\tilde{a}_{(I_-, I_+ + n]}). \end{aligned}$$

We have to prove that

$$(4) \quad \Gamma(\tilde{a}) = \Gamma(\tilde{b}).$$

By (1) we can choose

$$\tilde{x}^- \in \omega^-(a) \cap \omega^-(b), \quad \tilde{x}^+ \in \omega^+(a) \cap \omega^+(b).$$

We set

$$a = \tilde{\Phi}(\tilde{x}_{(I_- - L, I_-]}^-, \tilde{a}, \tilde{x}_{(I_+, I_+ + L]}^+), \quad b = \tilde{\Phi}(\tilde{x}_{(J_- - L, J_-]}^-, \tilde{b}, \tilde{x}_{(J_+, J_+ + L]}^+).$$

By Lemma 4.4 and by Property (B) of \tilde{X} (4) will follow, once it is shown that

$$(5) \quad \Gamma(a) = \Gamma(b).$$

We will show that

$$(6) \quad a_{(I_-, I_- + P + Q_n]} = b_{(J_-, J_- + P + Q_n]},$$

$$(7) \quad \Gamma^-(a) \subset \Gamma^-(b_{(J_-, J_+ - n]}),$$

$$(8) \quad \Gamma^-(b) \subset \Gamma^-(a_{(I_-, I_+ - n]}),$$

and

$$(9) \quad a_{(I_+ - P - Q_n, I_+]} = b_{(J_+ - P - Q_n, J_+]},$$

$$(10) \quad \Gamma^+(a) \subset \Gamma^+(b_{(J_-, J_+]}),$$

$$(11) \quad \Gamma^+(b) \subset \Gamma^+(a_{(I_-, I_+]}),$$

and use Property (c) of X to confirm (5). For (6) observe that by (2)

$$(5.2) \quad a_{(I_-, I_- + P + Q_n]} = \tilde{\Phi}(\tilde{x}_{(I_- - L, I_-]}, \tilde{a}_{(I_-, I_- + \tilde{P} + \tilde{Q}_n]}) = \\ \tilde{\Phi}(\tilde{x}_{(I_- - L, I_-]}, \tilde{b}_{(J_+ - \tilde{P} - \tilde{Q}_n, J_+]}) = b_{(J_-, J_- + P + Q_n]}.$$

To prove (7), let $u^- \in \Gamma^-(a)$. Then $\Phi(u^-) \in \Gamma^-(\tilde{a})$, which by (3) implies that $\Phi(u^-) \in \Gamma^-(\tilde{b}_{(I_-, I_+ - n]})$, which in turn implies that $u^- \in \Gamma^-(b_{(J_-, J_+ - n]})$. For (8) one has the symmetric argument, and for (9), (10) and (11) again the symmetric argument. \square

6. INSTANTANEOUS PRESENTATIONS

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ is right instantaneous if $\omega_1^+(\sigma) \neq \emptyset, \sigma \in \Sigma$, equivalently, if $\omega_\infty^+(\sigma) \neq \emptyset, \sigma \in \Sigma$. Left instantaneity is defined time symmetrically. This notion of left instantaneity was considered by Matsumoto in [M1, Section 4]. We say that a subshift is bi-instantaneous, if it is left and right instantaneous. We give an example of a topologically transitive sofic system that is left instantaneous but not right instantaneous. This example is a variation of an example that was used by Carlsen and Matsumoto in [CM]. Let $\Sigma = \{0, 1, \alpha, \beta\}$ and exclude from $\Sigma^{\mathbb{Z}}$ the points that contain one of the following words: $10, 11, \alpha 0^n 1 \beta, \beta 0^n 1 \alpha, n \in \mathbb{N}$. In this way one obtains a left instantaneous sofic system in which the words of the form $0^n 1, n \in \mathbb{N}$, do not have a future that is compatible with their entire past. By a product construction one can obtain examples of this type of topologically transitive sofic systems that are neither left nor right instantaneous.

Theorem 6.1. *A sofic system admits a bi-instantaneous presentation.*

Proof. For the construction of a bi-instantaneous presentation of a sofic system $X \subset \Sigma^{\mathbb{Z}}$ set

$$\mathcal{V} = \{\Gamma_\infty^+(x^-) : x^- \in X_{(-\infty, 0]}\},$$

and, denoting for an admissible word a of X by $\mathcal{V}(a)$ the set of $V \in \mathcal{V}$ that contain a sequence that starts with a , set

$$\tau_a(V) = \{y^+ \in X_{[1, \infty)} : (a, y^+) \in \mathcal{V}(a)\}, \quad V \in \mathcal{V}(a).$$

For a point $x \in X$ denote by $I^+(x)$ the $I \in \mathbb{N}$ that is the minimum of the $I \in \mathbb{N}$, such that there is an $i \in (1, I]$ such that

$$(1) \quad \tau_{(i, I]} \upharpoonright \mathcal{V}_{[1, i]} = \text{id},$$

and denote by $i^+(x) \in (1, I]$ the uniquely determined $i \in (1, I]$ such that (1) holds for $I = I^+(x)$. It is

$$(2) \quad I^+(S_X(x)) + 1 \geq I^+(x), \quad x \in X.$$

To see this, note that

$$\mathcal{V}_{[1, i^+(Sx)+1]} \subset \mathcal{V}_{(1, i^+(Sx)+1]}$$

and

$$\tau_{x_{(i+(Sx)+1, I+(Sx)+1)}} \upharpoonright \mathcal{V}_{(1, i+(Sx)+1]} = \text{id}$$

would imply that

$$\tau_{x_{(i+(Sx)+1, I+(Sx)+1)}} \upharpoonright \mathcal{V}_{(1, i+(Sx)+1]} = \text{id},$$

which is impossible by the definition of I^+ . Denote by z^+ the point in $X_{(I^+(x), \infty)}$ that carries the right infinite concatenation of the word $x_{x_{(i^+, I^+)}}$. It follows from $\mathcal{V}_{x_{[1, i^+(x)]}} = \mathcal{V}_{x_{[1, I^+(x)]}}$, that

$$(4) \quad z^+ \in \omega^+(x_{[1, I^+(x)]}).$$

With I^- and I^+ , z^- defined time symmetrically one has that

$$(5) \quad I^-(S_X(x)) - 1 \leq I^-(x), \quad x \in X,$$

and

$$(6) \quad z^- \in \omega^-(x_{[I^-(x), 0]}).$$

We set

$$\Xi(x) = (x_{[I^-(x), 0]}, 0, x_{[1, I^+(x)]}), \quad x \in X,$$

and with μ denoting a bound for $\{|I^-(x)|, I^+ : x \in X\}$ we define an embedding ξ of X into $(\mathcal{L}_\mu(X) \times \Sigma \times \mathcal{L}_\mu(X))^{\mathbb{Z}}$ by

$$\xi(x) = (\Xi(S^{-i}x))_{i \in \mathbb{Z}}, \quad x \in X.$$

Set

$$\Delta = \Xi(X), \quad Y = \xi(X).$$

We prove that Y is a bi-instantaneous presentation of X . For this let

$$(a(-), \sigma, a(+)) \in \Delta,$$

and let $x \in X$ be such that

$$\Xi(x) = (a(-), \sigma, a(+)).$$

By (2) and (5)

$$\Xi(S_X^{-1}((x_{(-\infty, I^+(x)]}, z^+(x))) \in \omega_1^+(a(-), \sigma, a(+)),$$

which confirms the right instantaneity of Y . The proof that Y is left instantaneous is time symmetric and uses (2) and (6). \square

We give an example of a semi-synchronizing (see [Kr2]) right-instantaneous non-sofic system that is not left instantaneous, but has a left instantaneous presentation. For this, take as alphabet the set

$$\Sigma = \{\mathbf{1}, \alpha_\lambda, \alpha_\rho, \beta_\lambda, \beta_\rho, \gamma_\lambda, \gamma_\rho\},$$

and view Σ as a generating set of \mathcal{D}_3 with relations

$$\alpha_\lambda \alpha_\rho = \beta_\lambda \beta_\rho = \gamma_\lambda \gamma_\rho = \mathbf{1}, \quad \alpha_\lambda \beta_\rho = \beta_\lambda \alpha_\rho = \alpha_\lambda \gamma_\rho = \gamma_\lambda \alpha_\rho = \beta_\lambda \gamma_\rho = \gamma_\lambda \beta_\rho = 0.$$

The Motzkin shift M (in this case M_3) is defined as the subshift in $\Sigma^{\mathbb{Z}}$ contains all $x \in \Sigma^{\mathbb{Z}}$ such that

$$\prod_{I_- \leq i < I_+} x_i \neq 0, \quad I_-, I_+ \in \mathbb{Z}, \quad I_- < I_+.$$

Intersect M_3 with the subshift of finite type that is obtained by excluding from $\Sigma^{\mathbb{Z}}$ all points that contain one of the words $\gamma_\lambda \gamma_\lambda, \alpha_\lambda \alpha_\lambda \gamma_\lambda, \beta_\lambda \beta_\lambda \gamma_\lambda$ to obtain a subshift X . There is a topological conjugacy of X onto a subshift \bar{X} that is given by a 3-block mapping Φ ,

$$\begin{aligned} \Phi(\alpha_\lambda \beta_\lambda \gamma_\lambda) &= \Phi(\beta_\lambda \alpha_\lambda \gamma_\lambda) = \gamma_\lambda \\ \Phi(\sigma \sigma' \sigma'') &= \sigma', \quad \sigma \sigma'' \notin \{\alpha_\lambda \gamma_\lambda, \beta_\lambda \gamma_\lambda\}. \end{aligned}$$

Whereas the subshift X is bi-instantaneous, the subshifts $\bar{X}^{\langle[0,n]\rangle}$, $n \in \mathbb{N}$, are right-instantaneous but not left-instantaneous: The words $\gamma_\lambda \gamma_\lambda \mathbf{1}^n$, $n \in \mathbb{N}$, do not have a past that is compatible with their entire future context.

For a subshift $X \subset \Sigma^\mathbb{Z}$, for $L \in \mathbb{Z}_+$, and for mappings

$$\Psi : X_{[-L,L]} \rightarrow X_{[1,L+1]}$$

we formulate a condition

$$(RIa) : \quad \Psi(a) \in \Gamma^+(x^-, a_{[-L,0]}), \quad a \in X_{[-L,L]}, x^- \in \Gamma^-(a).$$

If a mapping $\Psi : X_{[-L,L]} \rightarrow X_{[1,L+1]}$ satisfies condition (RIa) then one has for $a \in X_{[-2L,L]}$, that

$$(a_{(-2L,0]}, \Psi(a_{[-L,+L]})) \in X_{(-2L,L+1]},$$

and it is meaningful to impose on Ψ a further condition

$$(RIb) : \quad \Psi((a_{(-2L,0]}, \Psi(a_{[-L,+L]}))_{[i-L,i+L]}) = \Psi(a_{[i-L,i+L]}), \\ -2L < i \leq 0, a \in X_{(-2L,L+1]}.$$

We say that a mapping $\Psi : X_{[-L,L]} \rightarrow X_{[1,L+1]}$ that satisfies condition (RIa) and also satisfies condition (RIb) is an RI -mapping.

We say that a subshift with an RI -mapping has property RI and we say that a subshift with property RI has property BI , if its inverse also has an RI -mapping.

Theorem 6.2. *A subshift $X \subset \Sigma^\mathbb{Z}$ admits a right instantaneous presentation if and only if it has property RI . A subshift $X \subset \Sigma^\mathbb{Z}$ admits a bi-instantaneous presentation if and only if it has property BI .*

Proof. Consider the situation that there is given a right instantaneous subshift $\hat{X} \subset \hat{\Sigma}^\mathbb{Z}$ and a topological conjugacy φ of \hat{X} onto a subshift $X \subset \Sigma^\mathbb{Z}$ that is given by a one-block map $\Phi : \hat{\Sigma} \rightarrow \Sigma$ with φ^{-1} given for some $L \in \mathbb{N}$ by a block map

$$\hat{\Phi} : X_{[-L,L]} \rightarrow \hat{\Sigma}.$$

We choose a mapping $\hat{\Psi}$ that selects for every $\hat{\sigma} \in \hat{\Sigma}$ an element of $\hat{X}_{[1,L+1]} \cap \omega^+(\hat{\sigma})$, we set

$$\Psi = \Phi \hat{\Psi} \hat{\Phi},$$

and we show that Ψ is an IR -mapping for X . To see that Ψ satisfies condition (IRa) , let $a \in X_{[-L,L]}$ and let $x^- \in \Gamma^-(a)$. Then one has for

$$\hat{\sigma} = \hat{\Phi}(a), \quad \hat{x}^- = \hat{\Phi}(x^-, a_{[-L,L]}),$$

that

$$\hat{x}^- \in \Gamma^-(\hat{\sigma}),$$

and that

$$(1) \quad \Phi(\hat{x}^-, \hat{\sigma}, \hat{\Psi} \hat{\Phi}(\hat{\sigma})) = (\hat{x}, a_{[-L,0]}, \Psi(a)),$$

and therefore

$$\Psi(a) \in \Gamma^+(x^-, a_{[-L,0]}),$$

and condition (RIa) is shown. By (1) also

$$\Phi((a_{(-2L,0]}, \Psi(a_{[-L,+L]}))_{[i-L,i+L]}) = \Phi(a_{[i-L,i+L]}), \\ -2L < i \leq 0, a \in X_{(-2L,L+1]},$$

and this implies that Ψ satisfies condition (RIb) .

Conversely, let $X \subset \Sigma^\mathbb{Z}$ be a subshift with an IR -mapping $\Psi : X_{[-L,L]} \rightarrow X_{[1,L+1]}$. By RIa one obtains an embedding

$$\xi : X \rightarrow (\mathcal{L}_{[-L,L+1]}(X))^\mathbb{Z}.$$

by setting

$$\xi(x) = ((x_{[i-L, i]}, \Psi(x_{[i-L, i+L]})))_{i \in \mathbb{Z}}.$$

ξ is given by the block map

$$\Xi(a) = (a_{[-L, 0]}, \Psi(a)), \quad a \in X_{[-L, L]}.$$

Set

$$\Delta = \Xi(X_{[-L, L]}), \quad Y = \xi(X).$$

To prove that Y is right instantaneous we show for $a \in \Delta$ that $\Xi(a_{(-L, L+1]}) \in \omega^+(a)$. For this let $y^- \in \Gamma^-(a)$. Also let $x^- \in X_{(-\infty, L+1]}$ be such that

$$(y^-, a) = \Xi(x^-).$$

Set

$$z^- = (x_{(-\infty, 1]}^-, \Psi(a_{(-L, L+1]})).$$

By *RIa*

$$z^- \in X_{(-\infty, L+1]},$$

and by *RIb*

$$(y^-, a, \Xi(a_{(-L, L+1]})) = \Xi(z^-).$$

Assume that X is left instantaneous. To prove that Y is a then also left instantaneous we show for $a \in \Gamma$ and $\beta \in \omega_1^-(a_{-L})$ that $\Xi(\beta a_{[-L, L]}) \in \omega^-(a)$. For this let $y^+ \in \Gamma^+(a)$. Also let $x^+ \in X_{[-L, \infty)}$ be such that

$$(a, y^+) = \Xi(x^+).$$

One has by *RIb* that

$$\Xi(\beta x^+) = (\Xi(\beta a_{[-L, L]}), a, y^+).$$

Symmetry considerations complete the proof. \square

Theorem 6.2 suggests an alternate proof of Proposition 6.1.

Proposition 6.3. *Property RI and Property BI are invariants of topological conjugacy.*

Proof. Apply Theorem 6.2. \square

The coded system (see[BH]) with code

$$\mathcal{C} = \{0\alpha^n\beta^n : n \in \mathbb{N}\},$$

is an example of a synchronizing system that admits neither a left instantaneous presentation nor a right instantaneous presentation. We give an example of a semisynchronizing (see [Kr3]) non-synchronizing subshift that admits neither a left instantaneous presentation nor a right instantaneous presentation. For this, take as alphabet the set

$$\Sigma = \{\mathbf{1}, \alpha_\lambda, \alpha_\rho, \beta_\lambda, \beta_\rho\},$$

and view Σ as a generating set of \mathcal{D}_2 with relations

$$\alpha_\lambda \alpha_\rho = \beta_\lambda \beta_\rho = \mathbf{1}, \quad \alpha_\lambda \beta_\rho = \beta_\lambda \alpha_\rho = 0.$$

We let X be the subshift in $\Sigma^\mathbb{Z}$ that contains all $x \in M_2$ that are also label sequences of bi-infinite paths on the directed graph that has vertices $v, v(+), v(-)$, four loops at v , one with labels $\alpha_\lambda, \alpha_\rho, \beta_\lambda, \beta_\rho$, a loop at $v(-)$ with label β_λ , a loop at $v(+)$ with label β_ρ , an edge from v to $v(-)$ with label α_λ , an edge from $v(-)$ to $v(+)$ with label $\mathbf{1}$, and an edge from $v(+)$ to v with label α_ρ . Adding a loop at vertex $v(+)$ that carries the label $\mathbf{1}$ one obtains a semi-synchronizing non-synchronizing right instantaneous subshift that does not admit a left instantaneous presentation.

7. PRESENTING A CLASS OF SUBSHIFTS

We say that a bi-instantaneous subshift $X \subset \Sigma^{\mathbb{Z}}$ is strongly bi-instantaneous if the following holds:

For $N_o \in \mathbb{N}$ and $a^{(-)}, a^{(+)} \in \mathcal{L}(X)$ there exist $N \in \mathbb{N}$ and

$$c^{(-)} \in \omega^+(a^{(-)}), \quad c^{(+)} \in \omega^-(a^{(+)}),$$

together with a $c \in \Gamma^+(a^{(-)}c^{(-)}) \cap \Gamma^-(c^{(+)}a^{(+)})$, such that

$$N_o \leq \ell(c^{(-)}), \ell(c^{(+)}) < N,$$

and such that the word $a^{(-)}c^{(-)}c c^{(+)}a^{(+)}$ occurs in a point of $A_N(X)$.

We say that a subshift with Property *BI* has the strong Property *BI*, if it has a strongly bi-instantaneous presentation.

Proposition 7.1. *The strong (BI) property is an invariant of topological conjugacy.*

Proof. To prove the proposition it is enough to consider the situation that one is given a bi-instantaneous subshift $\widehat{X} \subset \widehat{\Sigma}^{\mathbb{Z}}$ and a strongly bi-instantaneous subshift $X \subset \Sigma^{\mathbb{Z}}$, and a topological conjugacy $\varphi : \widehat{X} \rightarrow X$ that is given by a one-block map $\widehat{\Phi} : \widehat{\Sigma} \rightarrow \Sigma$ with φ^{-1} given for some $L \in \mathbb{N}$ by a block map $\Phi : X_{[-L, L]} \rightarrow \widehat{\Sigma}$, and to prove that \widehat{X} is also strongly bi-instantaneous.

For this let $\widehat{a}^{(-)}, \widehat{a}^{(+)} \in \mathcal{L}(\widehat{X})$ and choose

$$\widehat{d}^{(-)} \in \Gamma_L^-(\widehat{a}^{(-)}), \quad \widehat{d}^{(+)} \in \Gamma_L^+(\widehat{a}^{(+)}), \quad \widehat{b}^{(-)} \in \omega_L^+(\widehat{a}^{(-)}), \quad \widehat{b}^{(+)} \in \omega_L^-(\widehat{a}^{(+)}),$$

and set

$$a^{(-)} = \widehat{\Phi}(\widehat{a}^{(-)}), \quad a^{(+)} = \widehat{\Phi}(\widehat{a}^{(+)}),$$

and

$$d^{(-)} = \widehat{\Phi}(\widehat{d}^{(-)}), \quad d^{(+)} = \widehat{\Phi}(\widehat{d}^{(+)}), \quad b^{(-)} = \widehat{\Phi}(\widehat{b}^{(-)}), \quad b^{(+)} = \widehat{\Phi}(\widehat{b}^{(+)}).$$

Set

$$N_o = \max(2L, \widehat{N}_o).$$

Because of the strong bi-instantaneity of X there is an $N \in \mathbb{N}$ and

$$c^{(-)} \in \omega^+(d^{(-)}a^{(-)}b^{(-)}), \quad c^{(+)} \in \omega^-(b^{(+)}a^{(+)}d^{(+)}),$$

together with a $c \in \Gamma^+(a^{(-)}c^{(-)}) \cap \Gamma^-(c^{(+)}a^{(+)})$, such that

$$N_o \leq \ell(c^{(-)}), \ell(c^{(+)}) < N,$$

and such that the word $d^{(-)}a^{(-)}b^{(-)}c^{(-)}c c^{(+)}b^{(+)}a^{(+)}d^{(+)}$ occurs in a point of $A_N(X)$.

We determine words $\widehat{c}^{(-)}, \widehat{c}, \widehat{c}^{(+)}$ by the conditions

$$\ell(\widehat{c}^{(-)}) = \ell(c^{(-)}), \quad \ell(\widehat{c}^{(+)}) = \ell(c^{(+)}),$$

and

$$\Phi(d^{(-)}a^{(-)}b^{(-)}c^{(-)}c c^{(+)}b^{(+)}a^{(+)}d^{(+)}) = \widehat{a}^{(-)}\widehat{c}^{(-)}\widehat{c} \widehat{c}^{(+)}\widehat{b}^{(+)}\widehat{a}^{(+)}.$$

We prove that

$$(1) \quad \widehat{c}^{(-)} \in \omega^+(\widehat{a}^{(-)}).$$

For this let

$$\widehat{w}^{(-)} \in \Gamma_{\infty}^-(\widehat{a}^{(-)}),$$

and set

$$w^{(-)} = \widehat{\Phi}(\widehat{w}^{(-)}).$$

We have

$$w^{(-)}a^{(-)}b^{(-)} = \widehat{\Phi}(\widehat{w}^{(-)}\widehat{a}^{(-)}\widehat{b}^{(-)}) \in \mathcal{L}(X),$$

and therefore also

$$w^{(-)}a^{(-)}b^{(-)}c^{(-)} \in \mathcal{L}(X),$$

which means that

$$\hat{c}^{(-)}\hat{a}^{(-)}\hat{c}^{(-)} = \Phi(w^{(-)}a^{(-)}b^{(-)}c^{(-)}) \in \mathcal{L}(\hat{X}),$$

and (1) is proved. A symmetric argument yields that also

$$(2) \quad \hat{c}^{(+)} \in \omega^{-}(\hat{a}^{(+)}).$$

If the word $d^{(-)}a^{(-)}b^{(-)}c^{(-)}c^{(+)}b^{(+)}a^{(+)}d^{(+)}$ occurs in a point $x \in X$ then the word

$$\hat{a}^{(-)}\hat{c}^{(-)}\hat{c}^{(+)}\hat{a}^{(+)} = \hat{\Phi}(d^{(-)}a^{(-)}b^{(-)}c^{(-)}c^{(+)}b^{(+)}a^{(+)}d^{(+)})$$

occurs in the point $\hat{x} = \varphi^{-1}(x) \in \hat{X}$. It is

$$(3) \quad \varphi^{-1}(A_N(X)) \subset A_N(X).$$

By (1), (2) and (3) the strong bi-instantaneity of \hat{X} will be shown. To recall the proof of (3) let

$$(4) \quad u \in A_N(X),$$

let $i \in \mathbb{Z}$, set $\hat{u} = \varphi^{-1}(u)$, and let

$$\hat{y}^{-} \in \Gamma^{-}(\hat{u}_{[i, i+N)}), \quad \hat{y}^{+} \in \Gamma^{+}(\hat{u}_{[i, i+N)}),$$

and have then

$$y^{-} = \hat{\Phi}(\hat{y}^{-}) \in \Gamma^{-}(\hat{u}_{[i, i+N)}), \quad y^{+} = \hat{\Phi}(\hat{y}^{+}) \in \Gamma^{+}(\hat{u}_{[i, i+N)}).$$

By (4) then

$$(y^{-}, u_{[i, i+N)}, y^{+}) \in X,$$

and one checks that

$$(\hat{y}^{-}, \hat{u}_{[i, i+N)}, \hat{y}^{+}) = \Phi(y^{-}, u_{[i, i+N)}, y^{+}). \quad \square$$

We note that we have also proved that for every bi-instantaneous presentation $X \subset \Sigma^{\mathbb{Z}}$ of a subshift with the strong *BI* property there is an n_o such that the presentations $X^{(n)}$, $n \geq n_o$, are strongly bi-instantaneous.

Inspection shows that for an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+})$, such that for $\mathfrak{r} \in \mathfrak{P}$ with only one predecessor vertex $\mathfrak{q} \in \mathcal{E}^{-}$, equivalently, with only one successor vertex $\mathfrak{q} \in \mathcal{E}^{+}$, one has that for $e^{-} \in \mathcal{E}^{-}(\mathfrak{q}, \mathfrak{r})$ there is an $e^{+} \in \mathcal{E}^{+}(\mathfrak{q}, \mathfrak{r})$, such that $(e^{-}, e^{+}) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})$, and also for $e^{+} \in \mathcal{E}^{+}(\mathfrak{q}, \mathfrak{r})$ there is an $e^{-} \in \mathcal{E}^{-}(\mathfrak{q}, \mathfrak{r})$, such that $(e^{-}, e^{+}) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})$, that an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+})$ -presentation $X(\mathcal{V}, \Sigma, \lambda)$ has Property (A), and that

$$\mathcal{S}(X(\mathcal{V}, \Sigma, \lambda)) = \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+}).$$

(Compare the case of the graph inverse semigroups of finite directed graphs, in which every vertex has at least two incoming vertices, that was considered in [HIK].) We adopt now this hypotheses on the structure of the \mathcal{R} -graph $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+})$ in order to ensure that a subshift with Property (A) exists, to which the \mathcal{R} -graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+})$ is associated.

Theorem 7.2. *For an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^{-}, \mathcal{E}^{+})$, such that for $\mathfrak{r} \in \mathfrak{P}$ with only one predecessor vertex $\mathfrak{q} \in \mathcal{E}^{-}$*

$$\mathcal{E}^{+}(\mathfrak{q}, \mathfrak{r}) \setminus \{e^{+} \in \mathcal{E}^{+}(\mathfrak{q}, \mathfrak{r}) : (e^{-}, e^{+}) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})\} \neq \emptyset, \quad e^{-} \in \mathcal{E}^{-}(\mathfrak{q}, \mathfrak{r}),$$

and

$$\mathcal{E}^{-}(\mathfrak{q}, \mathfrak{r}) \setminus \{e^{-} \in \mathcal{E}^{-}(\mathfrak{q}, \mathfrak{r}) : (e^{-}, e^{+}) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})\} \neq \emptyset, \quad e^{+} \in \mathcal{E}^{+}(\mathfrak{q}, \mathfrak{r}),$$

a subshift X with Property (A), to which there is associated the \mathcal{R} -graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, is topologically conjugate to an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation if and only if X has Property (C) and the strong BI property.

Proof. The hypothesis on $(\mathcal{G}_{\mathcal{R}}\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies that an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ presentation has property (C) and is strongly bi-instantaneous.

For the proof of the converse consider a subshift $X \subset \Gamma^{\mathbb{Z}}$ with Property (A) and associated semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, that has Property (C), and that is strongly bi-instantaneous. Also assume that X has Property (c) with respect to the parameter $(P, (Q_n)_{n \in \mathbb{Z}_+})$. Let

$$k_0 \geq P + Q_0,$$

be such that every element of $\mathfrak{P}(X)$ has a representative in $A_{k_0}(X)$. By Lemma

$$\omega^-(a) \cap \omega^+(a) \neq \emptyset, \quad a \in \mathcal{L}(X).$$

and by Property (c)

$$(1) \quad \Gamma(a) = \Gamma(aca), \quad c \in \omega^-(a) \cap \omega^+(a), a \in \mathcal{L}_k(X), k \geq k_0.$$

This allows to assign to a word $a \in \mathcal{L}_k(X), k \geq k_0$ a \approx -class $\mathfrak{p}(a) \in \mathfrak{P}(X)$ that contains the points that carry the bi-infinite concatenation of the word ac , where $c \in \omega^-(a) \cap \omega^+(a)$, since, as a consequence of (1), the class of these points does not depend on the choice of the word c .

For a word $a \in \mathcal{L}_{k_0+1}(X) \cup \mathcal{L}_{k_0+2}(X)$ we denote by $a^{(-)}$ the prefix of a that is obtained by removing the last symbol, and by $a^{(+)}$ the suffix of a that is obtained by removing the first symbol. For $a \in \mathcal{L}_{k_0+1}(X)$, let

$$c^{(-)} \in \omega^-(a^{(-)}) \cap \omega^+(a^{(-)}), \quad c^{(+)} \in \omega^-(a^{(+)}) \cap \omega^+(a^{(+)}),$$

and denote by $y[c^{(-)}, a, c^{(+)})$ the point $y \in Y(X)$, where $y_{[1, k_0+1]} = a$, and where $y_{(-\infty, 0]}$ carries the left infinite concatenation of the word $a^{(-)}c^{(-)}$ and $y_{((k_0+1), \infty)}$ carries the right infinite concatenation of the word $c^{(+)}a^{(+)}$.

Let η denote an isomorphism of $\mathcal{S}(X)$ onto $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. By (1) the image of the point $y[c^{(-)}, a, c^{(+)})$ under η does not depend on the choice of $c^{(-)}$ and $c^{(+)}$. As a consequence one obtains well defined mappings

$$f^{(-)} : \mathcal{L}_{k_0+1} \rightarrow \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+), \quad f^{(+)} : \mathcal{L}_{k_0+1} \rightarrow \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+),$$

by setting

$$\eta(y[c^{(-)}, a, c^{(+)}) = f^{(+)}(a)f^{(-)}(a), \quad a \in \mathcal{L}_{k_0+1}(X).$$

$X^{(k_0+2)}$ has the $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation $(\mathcal{V}, \Sigma, \lambda)$, where

$$\mathcal{V} = \mathcal{L}_{k_0+1}(X), \quad \Sigma = \mathcal{L}_{k_0+2}(X).$$

and

$$(1) \quad \begin{aligned} s(a) &= a^{(-)}, \quad t(a) = a^{(+)}, \\ \lambda(a) &= f^{(-)}(a^{(-)})f^{(+)}(a^{(+)}), \quad a \in \mathcal{L}_{k_0+2}(X). \end{aligned}$$

We set

$$(2) \quad \mathcal{V}(\mathfrak{p}) = \{a \in \mathcal{L}_{k_0+1}(X) : \mathfrak{p}(a) = \mathfrak{p}\}, \quad \mathfrak{p} \in \mathfrak{P}.$$

By (1) and (2) onw has (G1), (G2) and (G3) satisfied. That (G4) is satisfied is a consequence of the assumption that every vertex in \mathfrak{P} has at least two incoming edges in \mathcal{E}^- . The irreducibility of $(\mathcal{V}, \Sigma, \lambda)$ and (G5) follow from the surjectivity of η . \square

8. AN EXAMPLE

Let $\Pi > 1$. With the alphabet

$$\Sigma(\Pi) = \mathbb{Z}/\Pi\mathbb{Z} \cup \{\gamma^-(\delta_-, \beta_-), \gamma^+(\delta_+, \beta_+) : \delta_-, \beta_-, \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z}\}$$

we construct a subshift $X(\Pi) \subset \Sigma^{\mathbb{Z}}$ that has property (A) with associated semigroup $\mathcal{S}(\Pi)$ and that is not topologically conjugate to an $\mathcal{S}(\Pi)$ -presentation. We set

$$\eta(\gamma^-(\delta_-, \beta_-)) = -1, \eta(\tau) = 0, \eta(\gamma^+(\delta_+, \beta_+)) = 1, \quad \delta_-, \beta_-, \tau, \beta_+, \delta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

To obtain $X(\Pi)$ we exclude from $\Sigma(\Pi)^{\mathbb{Z}}$ the words

$$\tau\tau', \quad \tau, \tau' \in \mathbb{Z}/\Pi\mathbb{Z}.$$

We exclude the words

$$\tau w \tau', \quad \tau, \tau' \in \mathbb{Z}/\Pi\mathbb{Z}, \tau \neq \tau',$$

where $w = (\sigma_i)_{1 \leq i \leq I} \in \Sigma(\Pi)^{[1, I]}$, $I > 1$, $I \in \mathbb{N}$, is a word such that

$$\sigma_1 \in \{\gamma^+(\delta_+, \beta_+) : \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z}\}, \quad \sigma_I \in \{\gamma^+(\delta_+, \beta_+) : \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z}\},$$

and

$$\sum_{1 \leq i \leq I} \eta(\sigma_i) = 0.$$

We also exclude the words of length three in the symbols

$$\gamma^-(\delta_-, \beta_-), \quad \delta_-, \beta_- \in \mathbb{Z}/\Pi\mathbb{Z},$$

as well as the words of length three in the symbols

$$\gamma^+(\delta_+, \beta_+), \quad \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

and the words

$$\gamma^-(\delta_-, \beta_-)\tau\gamma^-(\delta'_-, \beta'_-), \quad \delta_-, \beta_-, \tau, \delta'_-, \beta'_- \in \mathbb{Z}/\Pi\mathbb{Z},$$

as well as the words

$$\gamma^+(\delta'_+, \beta'_+)\tau\gamma^+(\delta_+, \beta_+), \quad \delta'_+, \beta'_+, \tau, \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

We also exclude the words

$$\gamma^-(\delta_-, \beta_-)\gamma^+(\delta_+, \beta_+), \gamma^+(\delta_+, \beta_+)\gamma^-(\delta_-, \beta_-), \quad \delta_-, \beta_-, \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

and exclude the words

$$\gamma^-(\delta'_-, \beta'_-)\gamma^-(\delta_-, \beta_-)\tau$$

Also we exclude the words

$$\gamma^-(\delta_-, \beta_-)\tau\gamma^+(\delta_+, \beta_+), \quad \delta_-, \beta_-, \tau, \delta_+, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

where

$$\beta_- - \beta_+ \neq \tau,$$

and the words

$$\begin{aligned} \gamma^-(\delta'_-, \beta'_-)\gamma^-(\delta_-, \beta'_-)\tau\gamma^+(\beta'_+, \delta_+)\gamma^+(\beta_+, \delta'_+), \\ \delta'_-, \beta'_-, \delta_-, \beta'_-, \tau, \beta'_+, \delta_+, \beta_+, \delta'_+ \in \mathbb{Z}/\Pi\mathbb{Z}, \end{aligned}$$

where

$$\beta'_- - \beta'_+ = \tau, \quad \beta_- - \beta_+ \neq \delta_- - \delta_+.$$

Denote for $\delta_-, \beta'_- \in \mathbb{Z}/\Pi\mathbb{Z}$, by $\mathcal{W}^{(-)}(\delta_-, \beta'_-)$ the set of words $(\sigma_i)_{1 \leq i \leq I} \in \Sigma(\Pi)^{[1, I]}$, $I > 1$, $I \in \mathbb{N}$, such that

$$\sigma_1 = \gamma^-(\delta_-, \beta'_-), \quad \sum_{1 \leq i \leq I} \eta(\sigma_i) = 0,$$

and denote for $\beta'_+, \delta_+ \in \mathbb{Z}/\Pi\mathbb{Z}$, by $\mathcal{W}^{(+)}(\beta'_+, \delta_+)$ the set of words $(\sigma_i)_{1 \leq i \leq I} \in \Sigma(\Pi)^{[1, I]}$, $I > 1, I \in \mathbb{N}$, such that

$$\sum_{1 \leq i \leq I} \eta(\sigma_i) = 0, \quad \sigma_I = \gamma^+(\beta'_+, \delta_+).$$

We exclude the words

$$\gamma^-(\delta_-, \beta_-)w^{(-)}\tau w^{(+)}\gamma^+(\beta_+, \delta_+), \quad w^{(-)} \in \mathcal{W}^{(-)}(\delta_-, \beta'_-), w^{(+)} \in \mathcal{W}^{(+)}(\beta'_+, \delta_+),$$

$$\delta_-, \beta_-, \delta_-, \beta'_-, \tau, \beta'_+, \delta_+, \beta'_+, \delta_+ \in \mathbb{Z}/\Pi\mathbb{Z},$$

where

$$\delta_- - \delta_+ \neq \tau,$$

or where

$$\delta_- - \delta_+ = \tau, \quad \beta_- - \beta_+ \neq \tau.$$

Theorem 8.1. *The subshift $X(\Pi)$ has Property (A) with associated semigroup $\mathcal{S}(\Pi)$.*

Proof. Let $n \in \mathbb{N}, I_-, I_+ \in \mathbb{Z}$,

$$I_+ - I_- > 2(n+1),$$

and let

$$(1) \quad a \in X(\Pi)_{(I_-, I_+]} \in M_n(X(\Pi)).$$

We set

$$h(a) = - \min_{I_- \leq J_- \leq I_+} \left\{ \sum_{I_- < j \leq J_-} \eta(a_j) \right\},$$

noting, that also

$$h(a) = \max_{I_- \leq J_+ \leq I_+} \left\{ \sum_{J_+ < j \leq I_+} \eta(a_j) \right\}.$$

(1) implies that there are indices $J_-, J_+ \in \mathbb{Z}$,

$$I_- < J_- < n+1, \quad I_+ - n - 1 < J_+ < I_+,$$

that are given by

$$J_- = \min \left\{ J : - \sum_{I_- \leq j \leq J} \eta(a_j) = h(a), a_j \in \mathbb{Z}/\Pi\mathbb{Z} \right\},$$

$$J_+ = \max \left\{ J : \sum_{I_- \leq j \leq J} \eta(a_j) = h(a), a_j \in \mathbb{Z}/\Pi\mathbb{Z} \right\},$$

We set

$$Y^- = \bigcap_{Q \in \mathbb{N}} \{ y^- \in \Gamma(a_{I_-, I_+ + n + 1}) : \sum_{I_- - Q \leq i \leq I_-} \eta(a_i) \leq h(a) \},$$

$$Y^+ = \bigcap_{Q \in \mathbb{N}} \{ y^- \in \Gamma(a_{I_-, I_+ + n + 1}) : - \sum_{I_+ \leq i \leq I_+ + Q} \eta(a_i) \leq h(a) \},$$

and

$$Q^-(y^-) = \max \{ Q \in \mathbb{N} : \sum_{I_- - Q \leq i \leq I_-} \eta(a_i) = h(a) + 1 \}, \quad y^- \in \Gamma^-(a) \setminus Y^-,$$

$$Q^+(y^+) = \min \{ Q \in \mathbb{N} : - \sum_{I_+ \leq i \leq I_+ + Q} \eta(a_i) = h(a) + 1 \}, \quad y^- \in \Gamma^-(a) \setminus Y^-.$$

Here

$$y_{K_-}^-(y^-) \in \{ \gamma^-(\delta_-, \beta_-) : (\delta_-, \beta_- \in \mathbb{Z}/\Pi\mathbb{Z}) \},$$

$$y_{K_+}^-(y^+) \in \{ \gamma^+(\beta_+, \delta_+) : (\beta_+, \delta_+ \in \mathbb{Z}/\Pi\mathbb{Z}) \},$$

and we denote for $y^- \in \Gamma^-(a) \setminus Y^-$ by $\beta_-(y^-)$ the element of $\mathbb{Z}/\Pi\mathbb{Z}$ that appears as the β_- in $y_{K_-(y^-)}^-$, and for $y^- \in \Gamma^-(a) \setminus Y^-$ by $\beta_-(y^-)$ the element of $\mathbb{Z}/\Pi\mathbb{Z}$ that appears as the β^+ in $y_{K_+(y^+)}^+$. With this notation

$$\begin{aligned} \Gamma(a) = \\ (Y^- \times \Gamma^+(a)) \cup \{(y^- \in Y^-, y^+ \in Y^+) : \beta_+(y^-) - \beta_-(y^+) = \tau(a)\} \cup (\Gamma^-(a) \times Y^+) , \end{aligned}$$

which shows that $X(\Pi)$ has property (A).

To indicate how the semigroup, that is associated to $X(\Pi)$, arises, let $\mathbb{Z}/\Pi\mathbb{Z}$ be the vertex set of $G(\Pi)$, and denote the edges in $G(\Pi)$, that go from vertex τ to vertex τ' , by $e(\tau, \delta, \beta, \tau')$, $\tau, \delta, \beta, \tau' \in \mathbb{Z}/\Pi\mathbb{Z}$. A representative of the idempotent $\mathbf{1}_\tau \in \mathcal{S}(\mathcal{G}(\Pi))$, $\tau \in \mathbb{Z}/\Pi\mathbb{Z}$, is given, for example, by a point in $X(\Pi)$ that carries the bi-infinite concatenation of the word $\gamma^-(0, 0)0\gamma^+(0, 0)0$, and representatives of the generators $e^-(\tau, \delta, \beta, \tau')$ and $e^+(\tau, \delta, \beta, \tau')$ of $\mathcal{S}(\mathcal{G}(\Pi))$ are given, for example, respectively by the points $x_0^{\tau, \delta, \beta, \tau'}, x_0^{\tau', \beta, \delta, \tau} \in X(\Pi)$ where

$$x_0^{\tau, \delta, \beta, \tau'} = \gamma^-(\delta, \beta), \quad x_0^{\tau', \beta, \delta, \tau} = \gamma^+(\beta, \delta),$$

and where $x_{(0, \infty)}^{\tau, \delta, \beta, \tau'}$ and $x_{(0, \infty)}^{\tau', \beta, \delta, \tau}$ carry the right-infinite concatenation, and $x_{(-\infty, 0)}^{\tau, \delta, \beta, \tau'}$ and $x_{(-\infty, 0)}^{\tau', \beta, \delta, \tau}$ carry the left-infinite concatenation of the word $\gamma^-(0, 0)0\gamma^+(0, 0)0$. \square

Proposition 8.2. *The subshift $X(\Pi)$ does not have Property (C).*

Proof. Let $P \in \mathbb{N}$, and set

$$c_{P, \tau} = (0\gamma^-(0, 0)0\gamma^+(0, 0)^2)^P \tau (\gamma^-(0, 0)^2 0\gamma^+(0, 0)0)^P, \quad \tau \in \mathbb{Z}/\Pi\mathbb{Z}.$$

The left contexts of the words $c_{P, \tau}$, $\tau \in \mathbb{Z}/\Pi\mathbb{Z}$, are the same, their right contexts are the same, and for $\beta_-, \tau, \beta_+ \in \mathbb{Z}/\Pi\mathbb{Z}$ the pair

$$\begin{aligned} (\gamma^-(0, \beta_-)(\gamma^-(0, 0)^2 0\gamma^+(0, 0)0)^{P-1} \gamma^-(0, 0)^2 0\gamma^+(0, 0), \\ \gamma^-(0, 0)0\gamma^+(0, 0)^2 (0\gamma^-(0, 0)0\gamma^+(0, 0)^2)^{P-1} \gamma^+(\beta_+, 0)) \end{aligned}$$

is in the context of $c_{P, \tau}$ if and only if

$$\beta_- - \beta_+ = \tau. \quad \square$$

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9. MARKOV CODES AND ZETA FUNCTIONS

Denoting by $\Pi_n(X)$ the number of points of period n of a shift-invariant set $X \subset \Sigma^\mathbb{Z}$, the zeta function of X is given by

$$\zeta_X(z) = e^{\sum_{n \in \mathbb{N}} \frac{\Pi_n(X) z^n}{n}}.$$

We also recall from [Ke] the notion of a circular Markov code to the extent that is needed here. We let a Markov code be given by code \mathcal{C} of words in the symbols of a finite alphabet Σ together with a finite set \mathcal{V} and mappings $r : \mathcal{C} \rightarrow V, s : \mathcal{C} \rightarrow V$. To a Markov code (\mathcal{C}, r, s) there is associated the shift invariant set $X_{(\mathcal{C}, r, s)} \subset \Sigma^\mathbb{Z}$ of points $x \in \Sigma^\mathbb{Z}$ such that there are indices $I_k, k \in \mathbb{Z}$,

$$I_0 \leq 0 < I_1, \quad I_k < I_{k+1}, \quad k \in \mathbb{Z},$$

such that

$$(1) \quad x_{[I_k, I_{k+1})} \in \mathcal{C}, \quad k \in \mathbb{Z},$$

and

$$(2) \quad r(x_{[I_{k-1}, I_k)}) = s(x_{[I_k, I_{k+1})}), \quad k \in \mathbb{Z}.$$

(\mathcal{C}, r, s) is said to be a circular Markov code if for every periodic point x in $X_{(\mathcal{C}, r, s)}$ the indices $I_k, k \in \mathbb{Z}$, such that (1) and (2) hold, are uniquely determined by x . Given a circular Markov code (\mathcal{C}, s, r) denote by $\mathcal{C}(u, w)$ the set of words $c \in \mathcal{C}$ such that $s(c) = u, r(c) = w, u, w \in \mathcal{V}$. Set

$$g_{\mathcal{C}(u, v)} = \sum_{0 \leq n < \infty} \text{card}\{c \in \mathcal{C} : s(c) = u, r(c) = v, \ell(c) = n\},$$

and introduce the matrix

$$H^{(\mathcal{C})}(z) = (g_{\mathcal{C}(u, v)}(z))_{u, v \in \mathcal{V}}.$$

For a circular Markov code (\mathcal{C}, s, r) , one has [Ke]

$$(3) \quad \zeta_{X_{(\mathcal{C}, r, s)}}(z) = \det(I - H^{(\mathcal{C})}(z))^{-1}.$$

Given an \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. we associate to an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation $X(\mathcal{V}, \Sigma, \lambda)$ the state spaces

$$\Sigma^- = \{\sigma \in \Sigma : \lambda(\sigma) \in \mathcal{S}^- \cup \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}\},$$

and

$$\Sigma^+ = \{\sigma \in \Sigma : \lambda(\sigma) \in \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}^+\},$$

together with the 0 - 1 transition matrices $(A^-(\rho, \tau))_{\rho, \tau \in \Sigma^-}$ and $(A^+(\rho, \tau))_{\rho, \tau \in \Sigma^+}$, where for ρ, τ in $\Sigma^-(\Sigma^+)$ we set $A^-(\rho, \tau)(A^+(\rho, \tau))$ equal to 1 if and only if $r(\rho) = s(\tau)$. We denote the (possibly empty) topological Markov shift with state space $\Sigma^-(\Sigma^+)$ and transition matrix $A^-(A^+)$ by $X(\Sigma^-, A^-)(X(\Sigma^+, A^+))$. Also we associate to the finite directed labeled graph $(\mathcal{V}, \Sigma, \lambda)$ the circular Markov code $(\mathcal{C}^0(\mathcal{V}, \Sigma, \lambda), r, s)$, where $\mathcal{C}^0(\mathcal{V}, \Sigma, \lambda)$ ist the set of words

$$(\sigma_i)_{1 \leq i \leq I} \in \mathcal{L}(X(\mathcal{V}, \Sigma, \lambda)), \quad I > 1,$$

such that

$$\begin{aligned} \lambda((\sigma_i)_{1 \leq i \leq I}) &\in \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}, \\ \lambda((\sigma_i)_{1 \leq i \leq J}) &\notin \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}, \quad 1 < J < I, \end{aligned}$$

we let

$$\mathcal{C}_o^-(\mathcal{V}, \Sigma, \lambda) \quad (\mathcal{C}_o^+(\mathcal{V}, \Sigma, \lambda))$$

be the set of words

$$(\sigma_i)_{1 \leq i \leq I} \in \mathcal{L}(X(\mathcal{V}, \Sigma, \lambda)), \quad I > 1,$$

such that

$$\begin{aligned} \lambda((\sigma_i)_{1 \leq i \leq I}) &\in \mathcal{S}^- \cup \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}, \\ \lambda((\sigma_i)_{J \leq i \leq I}) &\in \mathcal{S}^+, \quad 1 < J \leq I, \\ (\lambda((\sigma_i)_{1 \leq i \leq I}) &\in \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}^+, \\ \lambda((\sigma_i)_{1 \leq i \leq J}) &\in \mathcal{S}^+, \quad 1 \leq J < I,) \end{aligned}$$

and we associate to the finite directed labelled graph $(\mathcal{V}, \Sigma, \lambda)$ the circular Markov codes $(\mathcal{C}^-(\mathcal{V}, \Sigma, \lambda), r, s)$ and $(\mathcal{C}^+(\mathcal{V}, \Sigma, \lambda), r, s)$, where $\mathcal{C}^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}^+(\mathcal{V}, \Sigma, \lambda))$ ist the set of words that contains the words that are in $\mathcal{C}_o^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}_o^+(\mathcal{V}, \Sigma, \lambda))$ or that are concatenations of a word in $\mathcal{C}_o^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}_o^+(\mathcal{V}, \Sigma, \lambda))$ and a word in $\mathcal{L}(\Sigma^-, A^-)(\mathcal{L}(\Sigma^+, A^+))$.

Theorem 9.1.

$$\begin{aligned} \zeta_{X(\mathcal{V}, \Sigma, \lambda)}(z) = \\ \frac{\det(\mathbf{1} - H^{\mathcal{C}^0(\mathcal{V}, \Sigma, \lambda)}(z))}{\det(\mathbf{1} - A^- z) \det(\mathbf{1} - H^{\mathcal{C}^-(\mathcal{V}, \Sigma, \lambda)}(z)) \det(\mathbf{1} - H^{\mathcal{C}^+(\mathcal{V}, \Sigma, \lambda)}(z)) \det(\mathbf{1} - A^+ z)}. \end{aligned}$$

Proof. Apply the formula for the zeta function of a topological Markov shift ([LM]), and note that $\det(\mathbf{1} - A^-z) = 1(\det(\mathbf{1} - A^+z) = 1)$ if $X(\Sigma^-, A^-)(X(\Sigma^+, A^+))$ is empty. Apply formula (3) and collect the contributions to the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$. \square

Following [Ke], special cases of Theorem 7.1 appeared in [I], [KM2] and [IK].

We denote for an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentation $X(\mathcal{V}, \Sigma, \lambda)$, and for $\mathfrak{p} \in \mathfrak{P}$ by $P_{\mathfrak{p}}(X(\mathcal{V}, \Sigma, \lambda))$ the set of periodic points of $X(\mathcal{V}, \Sigma, \lambda)$ that carry for some $V \in \mathcal{V}$ a bi-infinite concatenation of a path b from V to V such that $\lambda(b) = \mathbf{1}_{\mathfrak{p}}$.

Proposition 9.2. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, be an \mathcal{R} -graph such that for $\mathfrak{r} \in \mathfrak{P}$ with only one predecessor vertex $\mathfrak{q} \in \mathcal{E}^-$, equivalently, with only one successor vertex $\mathfrak{q} \in \mathcal{E}^+$, one has that for $e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r})$ there is an $e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r})$, such that $(e^-, e^+) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})$, and also for $e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r})$ there is an $e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r})$, such that $(e^-, e^+) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r})$. Let $X(\mathcal{V}, \Sigma, \lambda)$ and $X(\tilde{\mathcal{V}}, \tilde{\Sigma}, \tilde{\lambda})$ be topologically conjugate $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ -presentations. Then*

$$\prod_{\mathfrak{p} \in \mathfrak{P}} (\xi - \zeta_{P_{\mathfrak{p}}(X(\mathcal{V}, \Sigma, \lambda))}(z)) = \prod_{\mathfrak{p} \in \mathfrak{P}} (\xi - \zeta_{P_{\mathfrak{p}}(X(\tilde{\mathcal{V}}, \tilde{\Sigma}, \tilde{\lambda}))}(z)).$$

Proof. The hypothesis on the \mathcal{R} -graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies that

$$P(A(X(\mathcal{V}, \Sigma, \lambda))) = \bigcup_{\mathfrak{p} \in \mathfrak{P}} P_{\mathfrak{p}}(X(\mathcal{V}, \Sigma, \lambda)),$$

and, moreover, that this is in fact the partition of $P(A(X(\mathcal{V}, \Sigma, \lambda)))$ into its \approx -equivalence classes, which is invariant under topological conjugacy. \square

In the case of graph inverse semigroups of finite directed graphs, in which every vertex has at least two incoming edges, the set $P(A(X(\mathcal{V}, \Sigma, \lambda)))$ appeared in [HI, HIK] as the set of neutral periodic points. For the case of a Markov-Dyck shift X the coefficient sequence of the Taylor expansion of the coefficient of $\xi^{\text{card}(\mathfrak{P})-1}$ in $\prod_{\mathfrak{p} \in \mathfrak{P}} (\xi - \zeta_{P_{\mathfrak{p}}(X)}(z))$ was introduced in [M3] as a generalization of the Catalan numbers. (With the Catalan numbers $C_k = \frac{1}{n+1} \binom{2n}{n}$ this sequence is in the case of the Dyck shift $D_N, N > 1$, equal to $N^k C_k, k \in \mathbb{Z}_+.$)

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